المادة: Statics and Dynamics **المرحلة الاولى**

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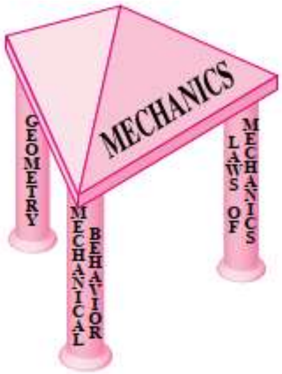
Lecture 1: Introduction to mechanics

❖ What is mechanics?

Mechanics is the study of force, deformation, and motion, and the relations between them.

Any mechanics problem can be divided into 3 parts which we think of as the 3 pillars that hold up the subject:

1. the mechanical behavior of objects and materials (constitutive laws);
2. the geometry of motion and distortion (kinematics); and
3. the laws of mechanics ($F = ma$, etc.).



1. Mechanical behavior

The first pillar of mechanics is mechanical behavior. The Mechanical behavior of something is the description of how loads cause deformation (or vice versa). When something carries a force it stretches, shortens, shears, bends, or breaks. Your finger tip squishes when you poke something.

This relation between force and deformation can be viewed in a few ways.

- First, it gives us a definition of force. E.g. force can be defined by the amount of spring stretch it causes.
- Second, different objects deform differently with the same loads implies. The relations of an object’s deformations to the forces that are applied are called the *mechanical properties* of the object. E.g. a piece of steel distorts under a given load differently than a same-sized piece of chewing gum. Mechanical properties are sometimes called *constitutive laws* because the mechanical properties describe how an object is constituted (at least from a mechanics point of view). The classic example of a constitutive law is that of a linear spring which you remember ‘$F = kx$’. When solving mechanics problems one has to make assumptions about the constitutive laws applicable to the parts of a system. How stretchy (elastic) or (viscous) or otherwise deformable is an object? The set of assumptions about the mechanical behavior of the system is sometimes called the constitutive model. Mechanics, where deformation is neglected, is called rigid body mechanics because a rigid (infinitely stiff) solid would not deform.

2. The geometry of deformation and motion

The second pillar of mechanics concerns the geometry of deformation and motion. Deformation is defined by changes of lengths and angles between sets of points. Motion is defined by the changes of the position of points in time. Concepts of length, angle, similar triangles, and the curves that particles follow and so on can be studied and understood without Newton’s laws and thus make up an independent pillar of the subject. Bicycles, planes, elevators, and hearses are designed to move people; a clock work, to move clock hands; insect wings, to move insect bodies.

The description of the motion of these things, of how the positions of the pieces change with time, of how the connections between pieces restrict the motion and the relations of these curves to each other is called *kinematics*. Kinematics is the study of the geometry of motion (or geometry in motion).

The geometry needed to understand small deformations and the geometry needed to understand large motions of rigid bodies are basic parts of mechanics.

3. Relation of force to motion, the laws of mechanics

The third pillar of mechanics is loosely called Newton's laws. One of Newton's brilliant insights was that the same 'force' that causes deformation also causes motion, or more precisely, acceleration of mass. Force is related to deformation by material properties (elasticity, viscosity, etc.) and to motion by the laws of mechanics, these are:

(1) The laws of mechanics apply to any system (rigid or not):
  a) Force and moment are the measure of mechanical interaction; and
  b) every action has an equal and opposite reaction.
(2) The net force on a system causes a net linear acceleration (linear momentum balance),
(3) The net turning effect of forces on system causes it to rotationally accelerate (angular momentum balance), and
(4) The change of energy of a system is due to the energy flow into the system (energy balance).

The principles of action and reaction, linear momentum balance, angular momentum balance, and energy balance, are actually redundant various ways. Linear momentum balance can be derived from angular momentum balance. Energy balance equations can often be derived from the momentum balance equations. The principle of action and reaction can also be derived from the momentum balance equations. In the practice of solving mechanics problems, however, the ideas are generally considered independently without much concern for which idea could be derived from the others for the problem under consideration. That is, the four assumptions above are not a mathematically minimal set, but they are all accepted truths in Newtonian mechanics.

❖ Statics, dynamics, and strength of materials

Elementary mechanics is traditionally divided into three courses named 'statics', 'dynamics', and 'strength of materials'. These subjects vary in how much they emphasize material properties, geometry, and Newton's laws. *Statics is mechanics with the idealization that the acceleration of mass is negligible in Newton's laws. Statics is generally about things that don't move much.* The first pillar of mechanics, constitutive laws, is generally introduced without fanfare into statics problems by the (implicit) assumption of rigidity. Other constitutive assumptions used include inextensible ropes, linear springs, and frictional contact. The material properties used as examples in elementary statics are very simple. Also, because things don't move or deform much in statics, the geometry of deformation and motion are all but ignored. Despite the commonly applied vast simplifications, statics is useful, for example, for the analysis of structures, slow machines or the light parts of fast machines, and the stability of boats.

Dynamics concerns motion associated with the non-negligible acceleration of mass. Dynamics thus concerns the two pillars the similar words kinematics and kinetics. Kinematics *concerns geometry without mentioning of force* and kinetics *concerns the relation of force to motion*. Once one has mastered statics, the hard part of dynamics is the kinematics. Dynamics is useful for the analysis of, for example, fast machines, vibrations, and ballistics.

Strength of materials expands statics to include material properties and also pays more attention to distributed forces (traction and stress). This course only occasionally touches lightly on strength of materials topics like stress (loosely, force per unit area), strain (a way to measure deformation), and linear elasticity (a commonly used constitutive model of solids). Strength of materials gives equal emphasis to all three pillars of mechanics. Strength of materials is useful for predicting the amount of deformation in a structure or machine and whether or not it is likely to break with a given load.

❖ Fundamental quantities in physics

- **Length**: the SI unit of length is **the meter** (abbreviated m) which is the distance that light travels in vacuum in (1 / 299,792,458) second, the speed of light in vacuum is $= 299792458\ m/s$.
- **Time**: the SI unit of time is the second (abbreviated s) It is based on an atomic clock, which uses the energy difference between the two lowest energy states of the cesium atom. When bombarded by microwaves of precisely the proper frequency, cesium atoms undergo a transition from one of these states to the other. One second (abbreviated s) is defined as the time required for 9,192,631,770 cycles of this microwave radiation.
- **Mass**: The standard of mass, the kilogram (abbreviated kg), is defined to be the mass of a particular cylinder of platinum–iridium alloy kept at the International Bureau of Weights and Measures at Sèvres, near Paris (Fig. 1.4). An atomic standard of mass would be more fundamental, but at present we cannot measure masses on an atomic scale with as much accuracy as on a macroscopic scale. The gram (which is not a fundamental unit) is 0.001 kilogram.

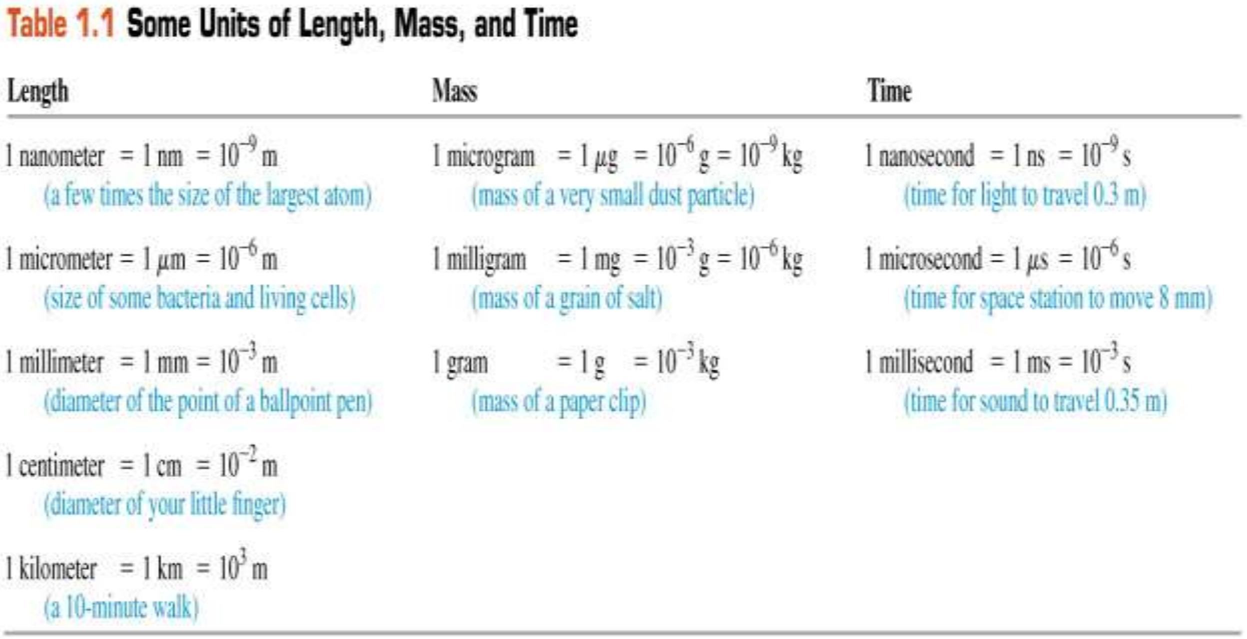


Table 1.1 Some Units of Length, Mass, and Time

Length	Mass	Time
1 nanometer = 1 nm = 10^{-9} m (a few times the size of the largest atom)	1 microgram = 1 μg = 10^{-6} g = 10^{-9} kg (mass of a very small dust particle)	1 nanosecond = 1 ns = 10^{-9} s (time for light to travel 0.3 m)
1 micrometer = 1 μm = 10^{-6} m (size of some bacteria and living cells)	1 milligram = 1 mg = 10^{-3} g = 10^{-6} kg (mass of a grain of salt)	1 microsecond = 1 μs = 10^{-6} s (time for space station to move 8 mm)
1 millimeter = 1 mm = 10^{-3} m (diameter of the point of a ballpoint pen)	1 gram = 1 g = 10^{-3} kg (mass of a paper clip)	1 millisecond = 1 ms = 10^{-3} s (time for sound to travel 0.35 m)
1 centimeter = 1 cm = 10^{-2} m (diameter of your little finger)		
1 kilometer = 1 km = 10^{3} m (a 10-minute walk)		

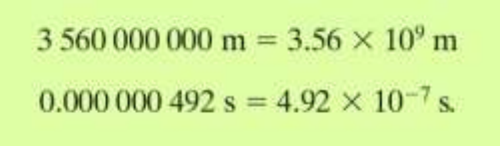
❖ Dimensions:

- **An equation must be** *dimensionally consistent*.
- **Terms to be added or equated must *always* have the same units. (Be sure you're adding "apples to apples.")**
- **Velocity x time = distance**
  - ***OK***: *5 meters/sec x 10 hours =~ 2 x* 10^{2} *km*

NOT: *5 meters/sec x 10 kg = 50 Joules :* (Velocity) x (mass) = (energy)

❖ Scientific Notation

- **Uses powers** of *10* to write large & small numbers



$$3\,560\,000\,000\ \text{m} = 3.56 \times 10^{9}\ \text{m}$$

$$0.000\,000\,492\ \text{s} = 4.92 \times 10^{-7}\ \text{s.}$$

❖ Accuracy & Error percentage

Accuracy is how close a measurement comes to the true value.

Example: Acceleration of Earth's gravity = $9.81\ m/sec^{2}$

Your experiment produces $10\ \ m/sec^{2}$

Were you accurate?

$$\%\ \text{ERROR} = \frac{|Actual - Measured|}{Actual} \times 100\ \% = \frac{|9.81 - 10|}{9.81} \times 100\% = 1.9$$

Lecture 2: Vectors: Scala and Vectors, Vector operations

❖ Vectors and Scalar

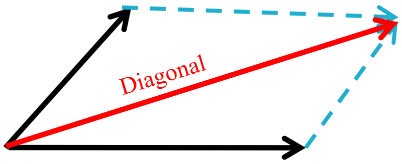
- **Scalar quantities** have size or magnitude, but a direction is not specified. (temperature, mass, speed, etc.)
- **Vector quantities** have magnitude and a specific direction (velocity, acceleration, etc.)

❖ The Resultant: The resultant is a vector that represents the sum of two or more vectors.

- The Parallelogram Law:

When two vectors are joined tail to tail

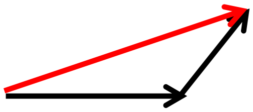
- Complete the parallelogram
- The resultant is found by drawing the diagonal line.



- The Triangle Law:

When two vectors are joined head to tail

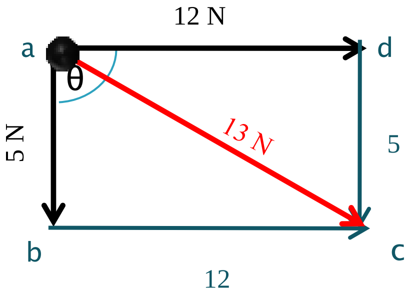
- Draw the resultant vector by completing the triangle



Example: Resultant of 2 Vectors

Two forces are applied to a body, as shown. What is the magnitude and direction of the resultant force acting on the body?

Solution:



- Complete the parallelogram (rectangle)
- The diagonal of the parallelogram *ac* represents the resultant force
- The magnitude of the resultant is found using Pythagoras' Theorem on the triangle *abc*

$$\text{Magnitude} = ac = \sqrt{12^2 + 5^2}$$

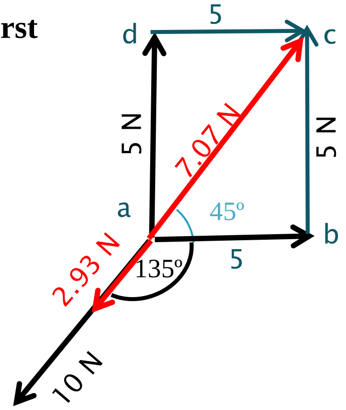
$$ac = 13\,\text{N}$$

$$\text{Directionof } ac : \tan\theta = \frac{12}{5}$$

$$\Rightarrow \theta = \tan^{-1}\frac{12}{5} = 67°$$

Example: Resultant of 3 Vectors

Find the magnitude and direction of the resultant of the three forces shown below



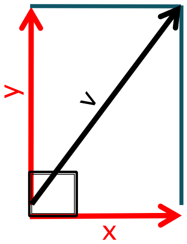
- **Find the resultant of the two 5 N forces first (do right angles first**

$$ac = \sqrt{5^2 + 5^2} = \sqrt{50} = 7.07\,\text{N}$$

$$\tan\theta = \frac{5}{5} = 1 \Rightarrow \theta = 45°$$

-
- Now find the resultant of the 10 N and 7.07 N forces
- The 2 forces are in a straight line (180° - 45° = 135°) and in opposite directions
- So, Resultant = 10 N – 7.07 N = 2.93 N in the direction of the 10 N force

❖ Resolving a Vector Into Perpendicular Components



- When resolving a vector into components we are doing the opposite to finding the resultant
- We usually resolve a vector into components that are perpendicular to each other
- Here a vector v is resolved into an x component and a y component

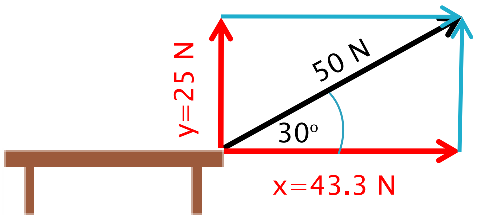
Practical Applications

Here we see a table being pulled by a force of 50 N at a 30° angle to the horizontal

When resolved we see that this is the same as pulling the table up with a force of 25 N and pulling it horizontally with a force of 43.3 N

- We can see that it would be more efficient to pull the table

with a horizontal force of 50 N

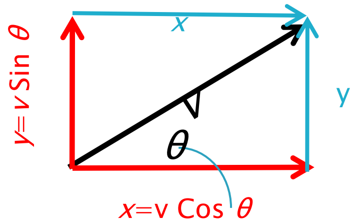


❖ Calculating the Magnitude of the Perpendicular Components

If a vector of magnitude *v* and makes an angle *θ* with the horizontal then the magnitude of the components are:

$x = v \cos \theta$

$y = v \sin \theta$



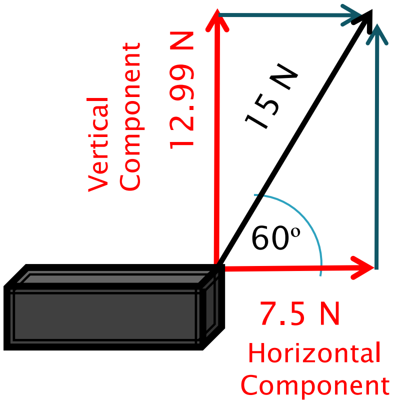
Proof:

$$Cos\theta = \frac{x}{v} \longrightarrow x = vCos\theta$$

$$Sin\theta = \frac{y}{v} \longrightarrow y = vSin\theta$$

Example: A force of 15 N acts on a box as shown.

What is the horizontal component of the force?

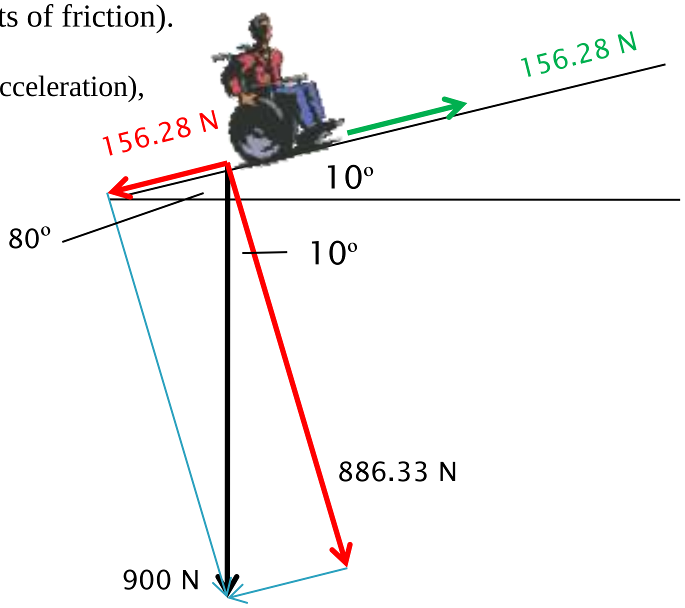


Solution:

$$\boxed{\text{Vertical Component} = y = 15Sin60° = 12.99\ \text{N}}$$

$$\boxed{\text{Horizontal Component} = x = 15Cos60° = 7.5\ \text{N}}$$

Example: A person in a wheelchair is moving up a ramp at constant speed. Their total weight is 900 N. The ramp makes an angle of 10° with the horizontal. Calculate the force required to keep the wheelchair moving at constant speed up the ramp. (You may ignore the effects of friction).



olution: If the wheelchair is moving at constant speed (no acceleration),

then the force that moves it up the ramp must be the same

as the component of it's weight parallel to the ramp.

- Complete the parallelogram.
- Component of weight parallel to ramp:

$$=900 Sin 10^\circ = 156.28\,\text{N}$$

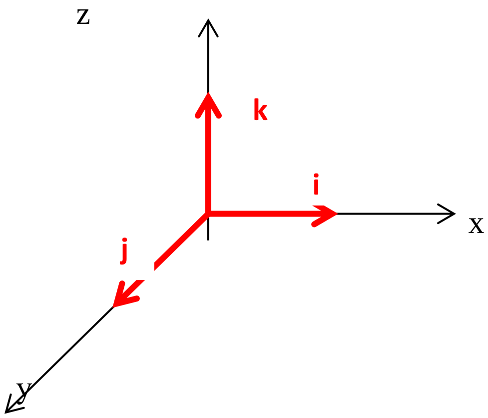
- Component of weight perpendicular to ramp:

$$=900 Cos 10^\circ = 886.33\,\text{N}$$

❖ Vector Multiplication

There are two ways of multiplying vectors

- ➢ Dot product (scalar product)
- ➢ Cross product (vector product)



- Coordinates: Unit Vectors

Cartesian coordinates: unit Vectors (i, j, k)

Any vector can be represented as $\mathbf{a} = a_1\boldsymbol{i} + a_2\boldsymbol{j} + a_3\boldsymbol{k}$

$\mathbf{b} = b_1\boldsymbol{i} + b_2\boldsymbol{j} + \boldsymbol{b_3 k}$

➢ **Dot Product:** Let *a* and *b* be two vectors, then the dot product of **a** and **b** is the scalar **a· b** given by

$$\boldsymbol{a.b} = a_1b_1 + a_2b_2 + a_3b_3 + \cdots + a_nb_n$$

$$|a| = \sqrt{{a_1}^2 + {a_2}^2 + {a_3}^2} \qquad |b| = \sqrt{{b_1}^2 + {b_2}^2 + {b_3}^2}$$

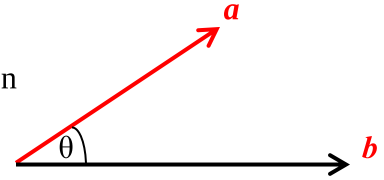
- **Properties of the Dot Product**

- If **a**, **b** are vectors and c is a scalar then
- 1) **a· a** $=|a|^2$
- 2) **a· b = b· a**
- 3) **a·** (**b**+ c) = **a· b**+ **a·** c
- 4) (c **a**) **·** **b** = c(**a· b**) = **a·** (c b)

- 5) 0· **a** = 0.

Theorem: If θ is the angle between the vectors ***a*** and ***b***, then

$$\boxed{\boldsymbol{a} \cdot \boldsymbol{b} = |a||b| \cos\theta}$$



NOTE: Two vectors **a** and **b** are orthogonal if and only if $\boldsymbol{a} \cdot \boldsymbol{b} = 0$

i.e. $\boldsymbol{\theta = 90}$

$$\therefore \boldsymbol{i} \cdot \boldsymbol{i} = \boldsymbol{j} \cdot \boldsymbol{j} = \boldsymbol{k} \cdot \boldsymbol{k} = (1)(1)\cos 0 = 1$$

$$\boldsymbol{i} \cdot \boldsymbol{j} = \boldsymbol{i} \cdot \boldsymbol{k} = \boldsymbol{j} \cdot \boldsymbol{k} = (1)(1)\cos 90^0 = 0$$

Example: find $\boldsymbol{u}.\boldsymbol{v}$, where

u = *3i* − 4*j* + *k* = *<3,-4,1>*

v = *5i* + 2*j* − 6*k*=*<5,2,-6>*

*Then find the angle formed by **u** and **v.***

Solution: Using the first method of calculation

$$u.v = (3)(5) + (-4)(2) + (1)(-6) = 15 - 8 - 6 = 1$$

To find the angle formed by u and v, use the second method of calculation. $a \cdot b = |a||b| \cos\theta$

$$|u| = \sqrt{9 + 16 + 1} = \sqrt{26}$$

$$|v| = \sqrt{25 + 4 + 36} = \sqrt{65}$$

$$\cos\theta = \frac{u.v}{|u||v|} = \frac{1}{\sqrt{26}\sqrt{65}} = \frac{1}{\sqrt{1690}} = 0.024$$

$$\theta = \cos^{-1}(0.024) = 88.6^o$$

Example 2: Given the vectors: u = 8i + 8j and v = —10i + 11j. find the following. $\boldsymbol{u} \cdot \boldsymbol{v}, \boldsymbol{v} \cdot \boldsymbol{u}, \boldsymbol{v} \cdot \boldsymbol{v}$

Solution:

$$\boldsymbol{u} \cdot \boldsymbol{v} = (8)(-10) + (8)(11) = -80 + 88 = 8$$

$$\boldsymbol{v} \cdot \boldsymbol{u} = (-10)(8) + (11)(8) = -80 + 88 = 8$$

$$\boldsymbol{v} \cdot \boldsymbol{v} = (8)(8) + (8)(8) = 64 + 64 = 128$$

Example 3: Given the vectors: ***u*** = 8i + 8j, ***v***= —10i + 11j, ***w*** = 9i + 7j.

Find the following. $\boldsymbol{u} \cdot (\boldsymbol{v} + \boldsymbol{w})$, $\boldsymbol{u} \cdot \boldsymbol{v} + \boldsymbol{u} \cdot \boldsymbol{w}$, $\boldsymbol{3(v \cdot u) - 12(w \cdot w)}$

Solution:

$$\boldsymbol{u}\cdot(\boldsymbol{v}+\boldsymbol{w}) = (8i+8j)\cdot[(-10i+11j)+(9i+7j)]$$

$$= (8i+8j)\cdot(-19i+18j) = (8)(-19)+(8)(18)$$

$$= -152+144 = -8$$

$$\boldsymbol{u}\cdot\boldsymbol{v}+\boldsymbol{u}\cdot\boldsymbol{w} = [(8)(-10)+(8)(8)]+[(8)(9)+(8)(7)]$$

$$= 8+128 = 136$$

$$3(\boldsymbol{v}\cdot\boldsymbol{u})-12(\boldsymbol{w}\cdot\boldsymbol{w}) = 3\boldsymbol{v}\cdot\boldsymbol{u}-12\boldsymbol{w}\cdot\boldsymbol{w}$$

$$= (-30i+33j)\cdot(8i+8j)-(27i+21j)\cdot(9i+7j)$$

$$= -240+264-(243+147) = 24+390 = 414$$

Example 4: Find the angle θ in degrees measured between the vectors

$$\boldsymbol{u} = 10i + 3j$$
$$\boldsymbol{v} = 1i — 7j$$

Solution: To find the angle formed by u and v, use $a\cdot b = |a||b|\,cos\,\theta$

$$\boldsymbol{u}.\boldsymbol{v} = (10)(1)+(3)(-7) = 10-21 = -11$$

$$|u| = \sqrt{100+9} = \sqrt{109} = 10.44$$

$$|v| = \sqrt{1+49} = \sqrt{50} = 7.071$$

$$\cos\theta = \frac{u.v}{|u||v|} = \frac{-11}{\sqrt{109}\sqrt{50}} = \frac{-11}{\sqrt{5450}} = -0.149$$

$$\theta = cos^{-1}(-0.149) = 98.57^{o}$$

Example 5: Determine if the pair of vectors is orthogonal, parallel, or neither. **u** = 7**i** – 3**j** and **v** = 21**i** – 9**j**

Solution: To the vectors, use $a\cdot b = |a||b|\,cos\,\theta$

$$\boldsymbol{u}.\boldsymbol{v} = (7)(21)+(-3)(-9) = 147+27 = 174$$

$$|u| = \sqrt{49+9} = \sqrt{58} = 7.616$$

$$|v| = \sqrt{441+81} = \sqrt{522} = 22.847$$

$$\cos\theta = \frac{u.v}{|u||v|} = \frac{174}{\sqrt{58}\sqrt{522}} = \frac{174}{\sqrt{30276}} = \frac{174}{174} = 1$$

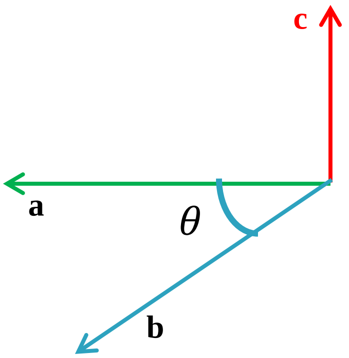
$$\theta = cos^{-1}(1) = 0$$

Since the angle between the vectors is zero. Therefore, the vectors are parallel.

- ## Cross Product
- Definition: If

$\mathbf{a} = a_1\boldsymbol{i} + a_2\boldsymbol{j} + a_3\boldsymbol{k}$

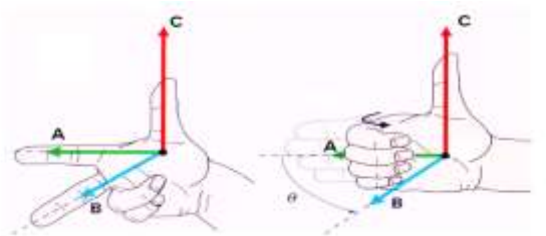
$\mathbf{b} = b_1\boldsymbol{i} + b_2\boldsymbol{j} + \boldsymbol{b_3k}$



then the cross product of **a** and **b** is the vector **c**

$$\boldsymbol{a} \times \boldsymbol{b} = \begin{vmatrix} i & j & k \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = c$$

$$c = (a_2b_3 - a_3b_2)i \ -(a_1b_3 - a_3b_1)j + (a_1b_2 - a_2b_1)k$$



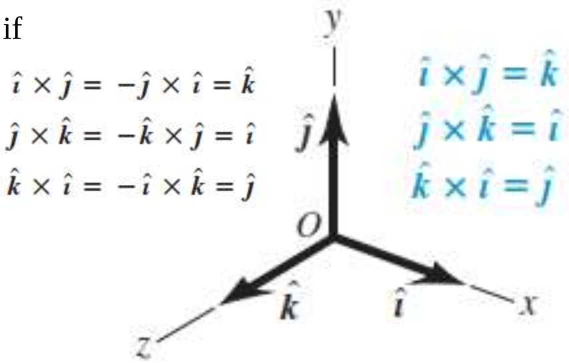
- **The direction of the cross product is perpendicular to *both* of the original vectors as determined by the**
  right hand rule.
- The vector $\boldsymbol{c} = \boldsymbol{a} \times \boldsymbol{b}$ is orthogonal to both $\boldsymbol{a}$ and $\boldsymbol{b}$.

If θ is the angle between $\boldsymbol{a}$ and $\boldsymbol{b}$, then

$$\boxed{|\mathbf{a} \times \mathbf{b}| = |\mathrm{a}||\mathrm{b}| \sin\theta}$$

- Two nonzero vectors **a** and **b** are **parallel** if and only if

$$\boxed{\boldsymbol{a} \times \boldsymbol{b} = 0}$$



- ## Properties of the Cross Product

If a and b and c are vectors and c is a scalar, then

1) $\boldsymbol{a} \times \boldsymbol{b} = -\boldsymbol{b} \times \boldsymbol{a}$

2) $(c\boldsymbol{a}) \times \boldsymbol{b} = c(\boldsymbol{a} \times \boldsymbol{b}) = \boldsymbol{a} \times (c\boldsymbol{b})$

3) $\boldsymbol{a} \times (\boldsymbol{b} + \boldsymbol{c}) = \boldsymbol{a} \times \boldsymbol{b} + \boldsymbol{a} \times \boldsymbol{c}$

4) $(\boldsymbol{a} + \boldsymbol{b}) \times \boldsymbol{c} = \boldsymbol{a} \times \boldsymbol{c} + \boldsymbol{b} \times \boldsymbol{c}$

5) $\boldsymbol{a}\cdot(\boldsymbol{b}\times\boldsymbol{c}) = (\boldsymbol{a}\times\boldsymbol{b})\cdot\boldsymbol{c}$

6) $\boldsymbol{a}\times(\boldsymbol{b}\times\boldsymbol{c}) = (\boldsymbol{a}\cdot\boldsymbol{c})\boldsymbol{b} - (\boldsymbol{a}\cdot\boldsymbol{b})\boldsymbol{c}$

Area and volume

- **The length of the cross product $\boldsymbol{a}\times\boldsymbol{b}$ is equal to the area of the parallelogram determined by a and b.**

- **The volume of the parallelepiped determined by the vectors a, b and c is the absolute value of the scalar triple product:**

V = |a· (b× c)|

Example 1: **Find u x v, where** $\boldsymbol{u} = 3i - 4j + k$ and $\boldsymbol{v} = 5i + 2j - 6k$

Solution

$$\boldsymbol{u}\times\boldsymbol{v} = \begin{vmatrix} i & j & k \\ 3 & -4 & 1 \\ 5 & 2 & -6 \end{vmatrix} = \boldsymbol{c}$$

$$\boldsymbol{c} = (24-2)i - (-18-5)j + (6+20)k = \mathbf{22}\boldsymbol{i} + \mathbf{23}\boldsymbol{j} + \mathbf{26}\boldsymbol{k}$$

Example 2: Vectors A and B have scalar product -6 and their vector product has magnitude 9. What is the angle between them?

Soltion:

$$\boldsymbol{A}\cdot\boldsymbol{B} = AB\cos\theta = -6$$
$$\boldsymbol{A}\times\boldsymbol{B} = AB\sin\theta = 9$$

$$\therefore \tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{9}{-6} = -1.5$$

$$\therefore \theta = \tan^{-1}(-1.5) = -56.31^0$$

Therefore the angle between the two vectors is $180^0 - 56.31^0 \approx 124^0$

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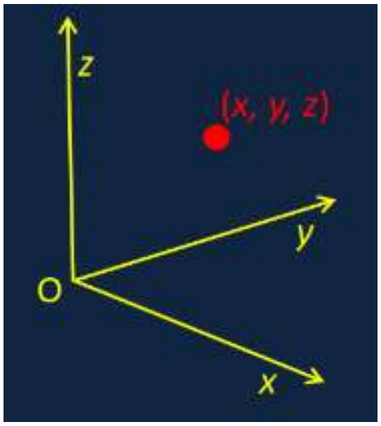
Lecture 3: Motion along a straight line

❖ Definitions

1. Distance: (scalar) is the total length that an object travels.
2. Displacement: (vector) is the length of space between the origin and the destination.
3. Speed: (scalar) the rate of movement of a particle.
   a. Average Speed = the ratio of the distance covered in a certain time
   b. Instantaneous Speed = how fast you are going at any particular instant. It is also known as the magnitude of the instantaneous velocity.
4. Velocity: (vector) the rate of displacement of a particle
   a. Average Velocity = the ratio of displacement covered in a certain time
   b. Instantaneous Velocity = the velocity of an object at an instant in time.
5. Acceleration: (vector) the rate at which velocity changes
   a. Average Acceleration = the ratio of velocity covered in a certain time
   b. Instantaneous Acceleration = the acceleration of an object at an instant in timed

❖ Frame of Reference

- To measure motion, we must first measure position.
- We measure position relative to some fixed point O, called the origin.
- We give the ball's location as (x, y, z): we reach it from O by moving x meters along the x-axis, followed by y parallel to the y-axis and finally z parallel to the z-axis.

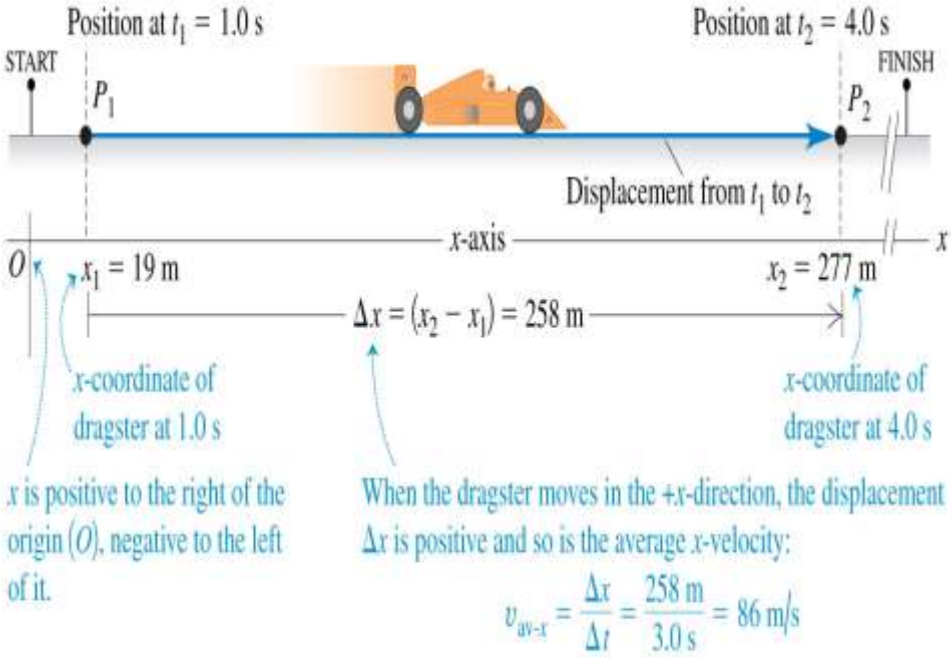


❖ Distance and Displacement

- ➢ **Distance** – the length of actual travel or how much an object has moved.
- ➢ **Displacement** – the length of a straight line drawn from the object's initial (starting) position to the object's final position

❖ Displacement, time, and average velocity

- A particle moving along the x-axis has a coordinate x.
- The change in the particle's coordinate is $\Delta x = x_2 - x_1$.
- The average x-velocity of the particle is $v_{av-x} = \Delta x / \Delta t$
- Figure 2.1 illustrates how these quantities are related.



❖ average speed and average velocity

The **average speed** of an object is defined as the total distance traveled divided by the total time elapsed.

$$\textit{Average speed} = \frac{\textit{total distance}}{\textit{total time}}$$

The **average velocity** is the rate at which the displacement occurs

$$\textit{Average velocity} = \frac{\Delta x}{\Delta t} = \frac{x_f - x_i}{t_f - t_i}$$

❖ Velocity can be positive or negative, but the time is always positive.

Example 1: Distance Run by a Jogger: How far does a jogger run in 1.5 hours (5400 s) if his average speed is 2.22 m/s?

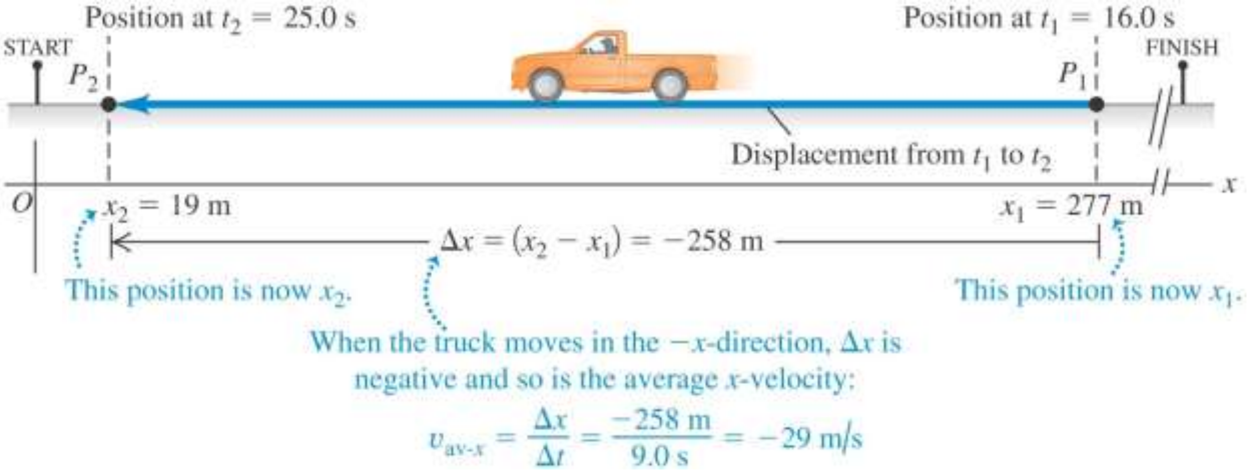
Solution:

$$\text{Average speed} = \frac{\text{Distance}}{\text{Elapsed time}}$$

$$\text{Distance} = (\text{Average speed})(\text{Elapsed time})$$

$$= (2.22 \text{ m/s})(5400 \text{s}) = 12000 \text{m}$$

❖ **Negative velocity:** The average x-velocity is *negative* during a time interval if the particle moves in the negative x-direction for that time interval. Figure 2.2 illustrates this situation.

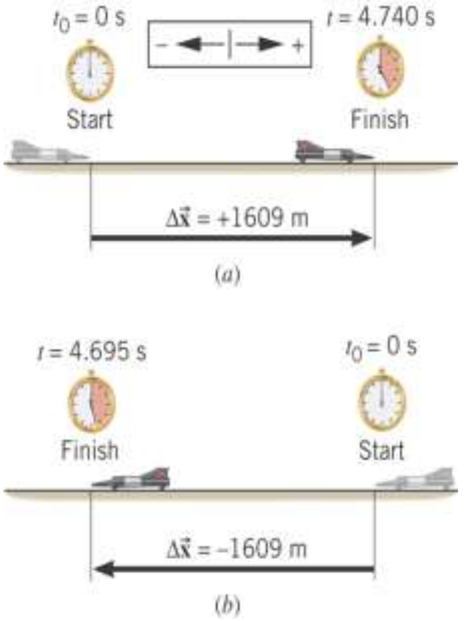


Example 2: The driver of the jet engine car makes two runs through the course, one in each direction, to nullify wind effects. From the data in the sketch, determine the average velocity for each run.

Solution:

$$v_{av} = \frac{\Delta x}{\Delta t} = \frac{+1609}{4.74} = +339.5\ \frac{m}{s}$$

$$v_{av} = \frac{\Delta x}{\Delta t} = \frac{-1609}{4.695} = -342.7\ \frac{m}{s}$$



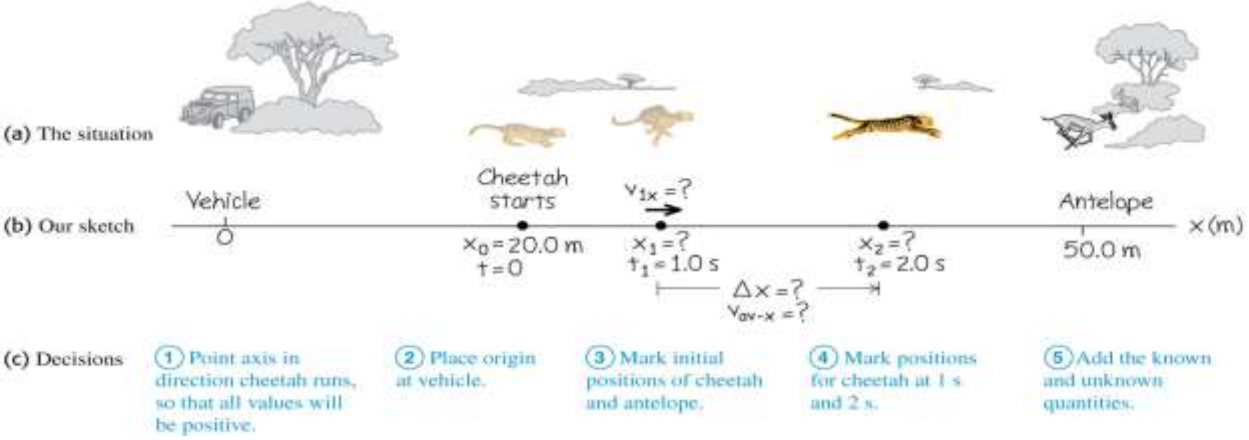
❖ Instantaneous velocity

The *instantaneous velocity* is the velocity at a specific instant of time and is given by the limit of the average velocity as the time interval becomes very short, or as the time interval approaches zero
(the derivative of x with respect to time).

$$v = \lim_{\Delta t \to 0} \frac{\Delta x}{\Delta t} = \frac{dx}{dt}$$

- The slope of the line tangent to the position-vs.-time graph is defined to be the instantaneous velocity at that time.
- The instantaneous speed is defined as the magnitude of the instantaneous velocity

❖ Average and instantaneous velocities



Example 3:

Example: Equation of motion

$$\boldsymbol{x = 20\,m + (5\ m/s^2)t^2}$$
$$\boldsymbol{x = x_0 \quad + \quad a \quad t^2}$$

(a) **Find the cheetah's displacement between $t_1 = 1s\ and\ t_2 = 2s$**
(b) **Find its average velocity during that interval.**
(c) **Find its instantaneous velocity at $t_1 = 1s$ by taking $\Delta t = 0.1\ s,\ then\ 0.01\ s,\ then\ 0.001\ s.$**
(d) **Derive an expression for the cheetah's instantaneous velocity as a function of time, and use it to find v_x at $t_1 = 1s$ and $t_1 = 2s$**

Solution:

(a) $x_1 = 20 + 5 \times 1^2 = 25\,m$, $x_2 = 20 + 5 \times 2^2 = 40\,m$

the displacement = $\Delta x = x_2 - x_1 = 40 - 25 = 15\,m$

(b) $\boldsymbol{v_{av} = \frac{\Delta x}{\Delta t} = \frac{15}{1} = 15\,m/s}$

(c) with $\Delta t = 0.1s$, the new $t_2 = 1.1\ s$, so the new position is

$$x_2 = 20 + 5\,(1.1)^2 = 26.05\,m,$$

$$v_{av} = \frac{\Delta x}{\Delta t} = \frac{26.05-20}{1.1-1} = 10.5\,m/s$$

$v_{av}\,(\Delta t = 0.01) = 10.05\,m/s$ ------ $v_{av}\,(\Delta t = 0.001) = 10.005\,m/s$

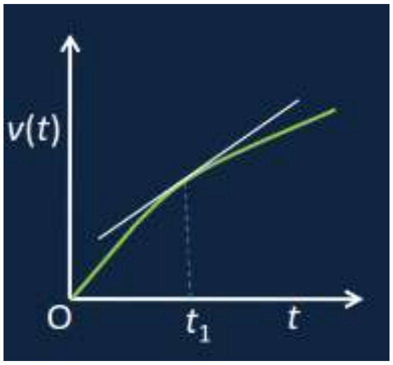
As Δt gets smaller, the average velocity gets closer to $10.0\,m/s$

Hence, the instantaneous velocity at $t_1 = 1s$ equals $10.0\,m/s$.

(d) $v_x = \frac{dx}{dt} = 0 + 5(2t) = 10\,t\,m/s^2$. Therefore,

$v_x(t = 1s) = 10\,(1) = 10\ m/s,\ v_x(t = 2s) = 10\,(2) = 20\ m/s$

❖ Average and instantaneous acceleration



- **Average acceleration** = $\boldsymbol{velocity\ change/time\ taken}$

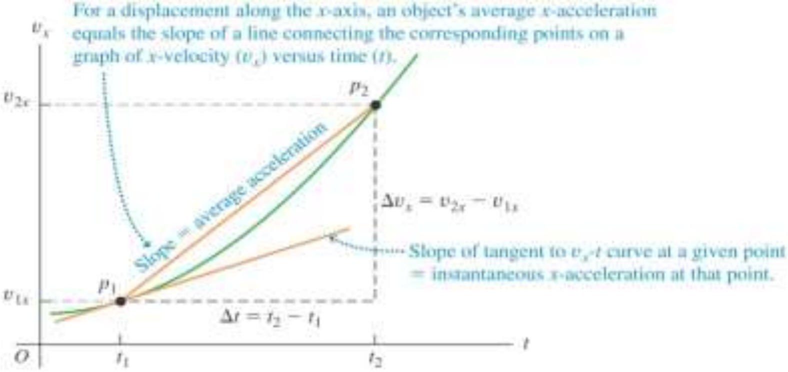
$$a_{av} = \frac{\boldsymbol{v_2 - v_1}}{t_1 - t_2} = \frac{\Delta v}{\Delta t}$$

  - Notice that ***acceleration*** relates to ***change in velocity*** exactly as ***velocity*** relates to ***change in displacement***.
- Velocity is a vector, so acceleration is a vector.
- **The instantaneous acceleration** is the slope of the tangent to the velocity vs time graph.

$$a = \lim_{\Delta t \to 0} \frac{\Delta v}{\Delta t} = \frac{dv}{dt}$$

❖ Average vs. instantaneous acceleration

- Acceleration describes the rate of change of velocity with time.
- The *average* x-acceleration is $a_{av-x} = \Delta v_x / Dt.$
- The *instantaneous* acceleration is $a_x = dv_x/\mathrm{dt}$



Example 4: A particle is traveling on a path given by the equation $\boldsymbol{x(t) = (t^2 - 4)(t + 5)}$ From 0 ≤ t ≤ 6 seconds.

Find the initial and final position, velocity and acceleration.

step 1: position

- $\boldsymbol{x(t) = t^3 + 5t^2 - 4t - 20}$
- $\boldsymbol{x(0) = -20\ m}$ $\boldsymbol{x(6) = 352\ m}$
- $\boldsymbol{x(t) = 0 \rightarrow (t^2 - 4)(t + 5) = 0 \rightarrow either\ \ t = \sqrt{4} = \mp 2\ \ or\ t = -5}$
- **The time can not be negative. Therefore, the particle will be at the origin at $\boldsymbol{t = 2\ s}$**

step 2: velocity

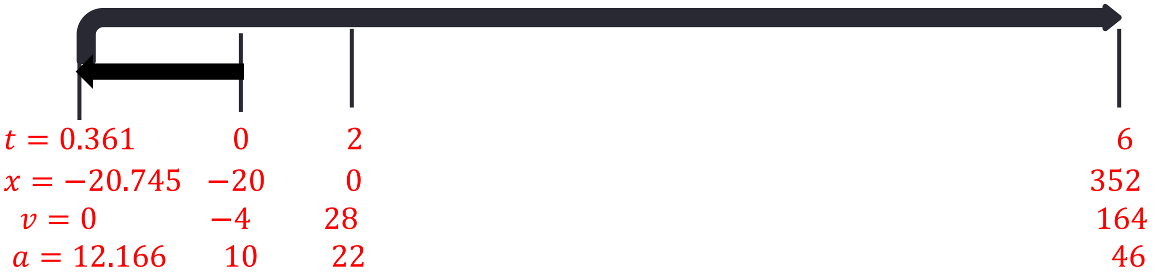
- $\boldsymbol{v(t) = \frac{dx}{dt} = 3t^2 + 10t - 4}$
- $\boldsymbol{v(0) = -4\,m/s}$ $\boldsymbol{v(6) = 164\,m/s}$
- $\boldsymbol{v(t) = 0 \rightarrow 3t^2 + 10t - 4 = 0 \rightarrow t = 0.361\ s}$
- $\boldsymbol{x(0.361) = (0.361)^3 + 5(0.361)^2 - 4(0.361) - 20 = -20.745\,m}$

This tells us our problem will be broken down into **two stages**, the one with the negative velocity (up to t=0.361) and the one with the positive velocity (from t=0.361 and beyond).

step 3: acceleration

$$\boldsymbol{a(t) = \frac{dv}{dt} = 6t + 10}$$

$$\boldsymbol{a(0) = 10\ m/s^2 \qquad a(6) = 46\,m/s^2}$$



❖ Constant Acceleration

Constant acceleration means the rate of change of velocity is constant.

$$\frac{dv}{dt} = a = constant$$

The solution to this equation is by integration

$$\int_{v_0}^{v} dv = \int a\,dt \quad \rightarrow \quad v = v_0 + at \qquad (1)$$

Example 1: a car traveling at $10m/s$ accelerates steadily at $2m/s^2$. How fast is it going after 2 secs? After 4 secs.

Solution: $v_0 = 10m/s$, $= \ 2m/s^2$, $t = 2s$, $t = 4s$.

1. $at\ t = 2s$: $\quad v = 10 + 2(2) = 14$ m/s
2. $at\ t = 4s$: $\quad v = 10 + 2(4) = \ 18$ m/s

❖ Distance Moved at Constant Acceleration

At constant acceleration,

$$\frac{dx}{dt} = v(t) = v_0 + at$$

The solution of this equation is by integration

$$\int_{x_0}^{x} dx = \int (v_0 + at)\,dt \quad \rightarrow \quad x(t) = x_0 + v_0 t + \frac{1}{2}at^2 \qquad (2)$$

Here x_0 is the start position, v_0 the start velocity, a the constant acceleration.

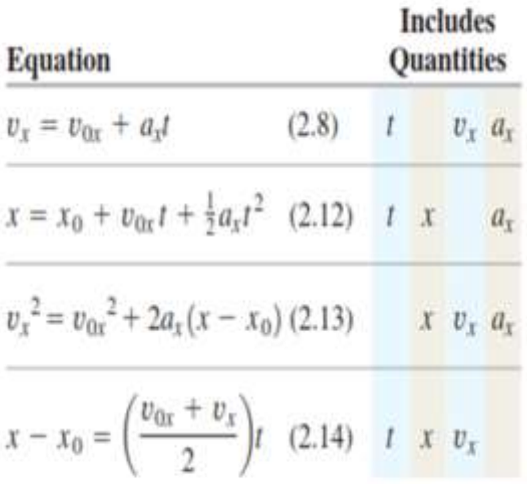
Example 2: A car starting from rest moving at velocity $10m/s$ and acceleration of $2m/s^2$. calculate the distance traveled after 5 s.

Solution:

$$\boldsymbol{x = 0 + 10\,(5) + \frac{1}{2} \times 2(5^2) = 50 + 25 = 75\ m.}$$

❖ The equations of motion with constant acceleration

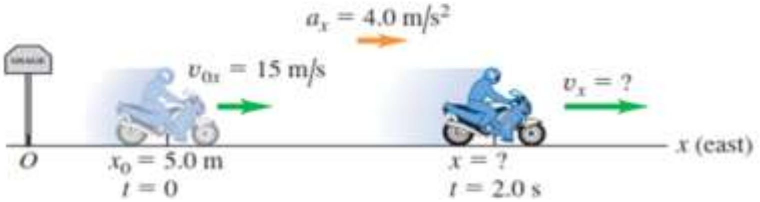
- $v = v_0 + at \qquad (1)$
- $x(t) = x_0 + v_0 t + \frac{1}{2}at^2 \qquad (2)$
  - ➢ Eliminating a from equations (1) and (2) results in
- $x - x_0 = \left(\frac{v_0 + v}{2}\right) t \qquad (3)$
  - ➢ Eliminating t from equations (1) and (2) results in
- $v^2 = {v_0}^2 + 2a(x - x_0) \qquad (4)$



Equation		Includes Quantities			
$v_x = v_{0x} + a_x t$	(2.8)	t		v_x	a_x
$x = x_0 + v_{0x} t + \frac{1}{2} a_x t^2$	(2.12)	t	x		a_x
${v_x}^2 = {v_{0x}}^2 + 2a_x(x - x_0)$	(2.13)		x	v_x	a_x
$x - x_0 = \left(\frac{v_{0x} + v_x}{2}\right) t$	(2.14)	t	x	v_x	

Example 3: A motorcyclist heading east through a small town accelerates at a constant $4\ m/s^2$ after he leaves the city limits. At time $t = 0$ he is 5.0 m east of the city-limits signpost, moving east at 15 m/s

- (a) Find his position and velocity at $t = 2\,s$.
- (b) Where is he when his velocity is $25\,m/s$?



Solution: $x_0 = 5m,\ v_0 = 15m/s,\ a = 4m/s^2$

a) Using equation (2) for the position

$$x(t) = x_0 + v_0 t + \frac{1}{2} a t^2$$

$$x(t = 2s) = 5 + 15 \times 2 + \frac{1}{2} \times 4 \times (2)^2 = 5 + 30 + 8 = 43\ m$$

Using equation (1) for the velocity $v = v_0 + at$

$$v = 15 + 4 \times 2 = 15 + 8 = 23\ m/s$$

b) To find the position when the velocity = 25 m/s, we should use equation (4). $v^2 = {v_0}^2 + 2a(x - x_0)$

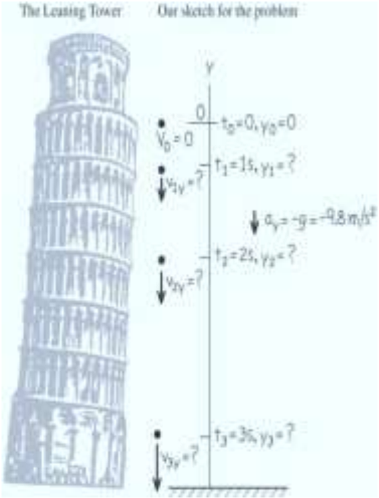
$$x = x_0 + \frac{v^2 - {v_0}^2}{2a}$$

$$x = 15 + \frac{25^2 - 15^2}{2 \times 4} = 55m$$

❖ Freely falling bodies

- *Free fall* is the motion of an object under the influence of only gravity.

Aristotle thought that heavy bodies fall faster than light ones, but Galileo showed that all bodies fall at the same rate. If there is no air resistance, the downward acceleration of any freely falling object is g = 9.8 m/s2 = 32 ft/s2. Follow Example 2.6 for a coin dropped from the Leaning Tower of Pisa.



Example: A one-euro coin is dropped from the Leaning Tower of Pisa and falls freely from rest. What are its position and velocity after 1s, 2 s, and 3 s?

- Given information: $y_0 = 0,\ v_0 = 0,\ t_0 = 0,\ a = -g$

  Position: $y = y_0 + v_0 t + \frac{1}{2}(-g)t^2$

$$\boldsymbol{y = 0 + 0 \times t + \frac{1}{2}(-g)t^2 = \frac{1}{2}(-9.8)t^2}$$

  Velocity: $v = v_0 + at$

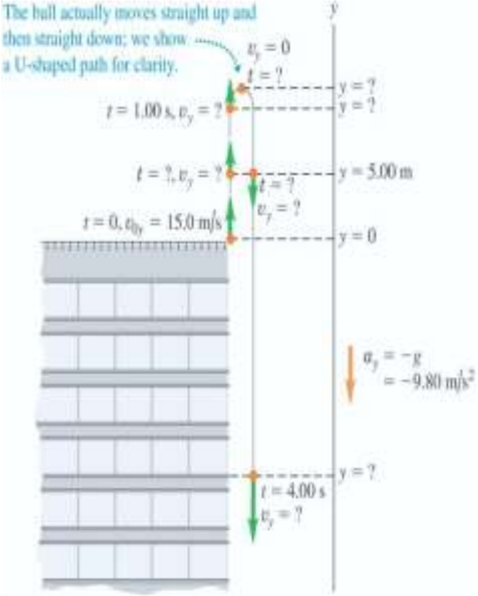
$$\boldsymbol{v = 0 + (-g)t = -9.8\,t}$$

❖ Up-and-down motion in free fall

Example: A ball is thrown vertically upward from the roof of a tall building. The ball leaves your hand at an upward speed of 15 m/s the ball is then in free fall. On its way back down, it just misses the railing. Find

(a) the ball's position and velocity 1s and 4s after leaving your hand;

(b) the ball's velocity when it is 5 m above the railing



(c) the maximum height reached;

(d) the ball's acceleration when it is at its maximum height

- Given information:

$$y_0 = 0,\ v_0 = 15\,m/s,\ t_0 = 0,\ a = -g$$

(a) Position: $y = y_0 + v_0 t + \frac{1}{2}(-g)t^2$

$$y(t = 1s) = 0 + 15 \times t + \frac{1}{2}(-g)t^2 = 15(1) + \frac{1}{2}(-9.8)(1)^2$$
$$= +10.1\,m$$
$$y(t = 4s) = 0 + 15 \times t + \frac{1}{2}(-g)t^2 = 15(4) + \frac{1}{2}(-9.8)(4)^2$$
$$= -18.4$$

(b) Velocity: $v = v_0 + at$

$$v(t = 1) = 15 + (-g)t = 15 - 9.8\,(1) = +5.2\,m/s$$
$$v(t = 4) = 15 + (-g)t = 15 - 9.8\,(4) = -24.2\,m/s$$

At $t = 1s$: the position and velocity are positive means that the ball above the origin (roof of the building)

At $t = 4s$: the ball is $\mathbf{18.4\,m}$ below the origin and moving downwards with a speed of $\mathbf{24.2\,m/s}$.

(c) At the instant at which the ball reaches its maximum height its y-velocity is momentarily zero: We use Eq. (4) $v^2 = {v_0}^2 + 2a(x - x_0)$ to find the maximum height.

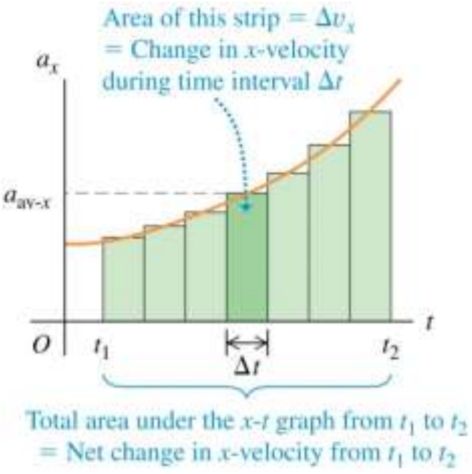
$$v_y = 0,\quad y_0 = 0,\quad a_y = -g$$
$$0 = {v_0}^2 + 2(-g)(y - y_0)$$
$$y = \frac{{v_0}^2}{2g} = \frac{15}{2(9.8)} = +11.5\,m$$

(e) At the highest point of free-fall motion, where the velocity is zero, the acceleration is NOT zero. Remember that acceleration is the rate of change of velocity, and the ball's velocity is continuously changing. At every point, including the highest point, and at any velocity, including zero, the acceleration in free fall is always $a_y = -g = -9.8\,m/s^2$

❖ Velocity and position by integration

- The acceleration of a car is not always constant.
- The motion may be integrated over many small time intervals to give

$$v_x = v_{ox} + \int_0^t a_x dt \quad \text{and} \quad x = x_0 + \int_0^t v_x dt.$$



Lecture 4: Motion in two dimensions

❖ Position and Velocity Vectors

➢ The position vector

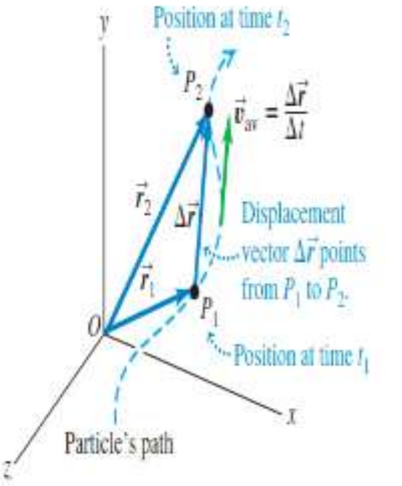
$$\boldsymbol{r} = x\boldsymbol{i} + y\boldsymbol{j}$$

During a time interval Δt the particle moves from P_1, where its position vector is r_1 to P_2 where its position vector is r_2 . The change in position (the displacement) during this interval is

$$\Delta \boldsymbol{r} = (x_2 - x_1)\boldsymbol{i} + (y_2 - y_1)\boldsymbol{j}$$

➢ The average velocity

$$\boldsymbol{v}_{av} = \frac{\Delta \boldsymbol{r}}{\Delta \boldsymbol{t}} = \frac{\boldsymbol{r}_2 - \boldsymbol{r}_1}{\boldsymbol{t}_2 - \boldsymbol{t}_1}$$



❖ Instantaneous velocity

Instantaneous velocity: It is the limit of the average velocity as the time interval approaches zero, and it equals the instantaneous rate of change of position with time.

$$\boldsymbol{v} = \lim_{\Delta t \to 0} \frac{\Delta r}{\Delta t} = \frac{dr}{dt}$$

Note that as $\Delta t \to 0$, points P_1 and P_2 move closer and closer together. In this limit, the vector $\Delta \boldsymbol{r}$ becomes tangent to the path.

$$\boldsymbol{v} = \quad v_x i + v_y j$$

$$\boldsymbol{v} = \frac{d\boldsymbol{r}}{dt} = \frac{dx}{dt} i + \frac{dy}{dt} j$$

❖ Magnitude & Direction of the instantaneous velocity:

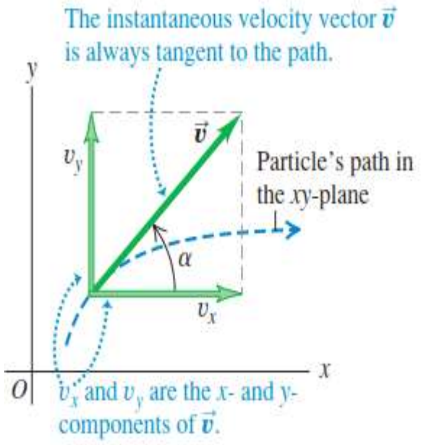
Magnitude = speed:

$$|v| = \sqrt{{v_x}^2 + {v_y}^2}$$

Direction of the instantaneous velocity:

$$\tan(\alpha) = \frac{v_y}{v_x}$$

$$\alpha = \arctan \frac{v_y}{v_x}$$



❖ Calculating average and instantaneous velocity

Example 1: an object is moving in the x-y plane. Its coordinates as a function of time are:

$$x = 2 - 0.25\, t^2$$

$$y = t + 0.025\, t^3$$

Find:

(a) its coordinates and distance from the origin at $t = 2\ s$.

(b) its displacement and average velocity vectors for the interval $t = 0\ to\ t = 2\ s$.

(c) a general expression for the instantaneous velocity vector $\boldsymbol{v}$. Express $v\ at\ t = 2s$ in component form and in terms of magnitude and direction.

Solution:

(a) Its coordinates at $t = 2s$ are:

$$x = 2 - 0.25\ (2)^2 = 2 - 1 = 1m$$
$$y = 2 + 0.025\ (2)^3 = 2 + 0.2 = 2.2m$$

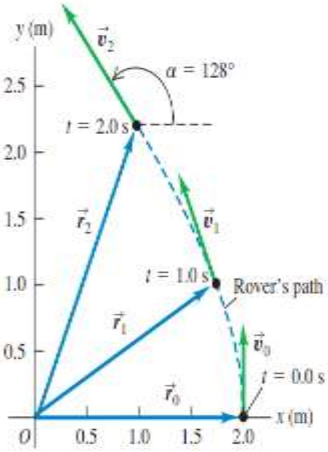
Its distance from the origin is

$$r = \sqrt{x^2 + y^2} = \sqrt{1^2 + (2.2)^2} = 2.4m$$

(b) Displacement $\Delta r = r_f - r_i$

Position vector $r = xi + yj$

$$r = (2 - 0.25\ t^2)i + (t + 0.025\ t^3)j$$
$$At\ t = 0\ ,\ r_0 = 2i + 0j = 2i = r_i$$
$$At\ t = 2,\ \ r_2 = i + 2.2j = r_f$$



$\Delta r = i - 2.2j - 2i = -i + 2.2j$

The average velocity $\quad v_{av} = \frac{\Delta r}{\Delta t} = \frac{-i+2.2j}{2-1} = -0.5i + 1.1j$

(c) Instantaneous velocity vector $\boldsymbol{v = v_x i + v_y j}$

$$v_x = \frac{dx}{dt} = \frac{d(2 - 0.25\ t^2)}{dt} = 0 - 0.25(2t) = -0.5\ t$$
$$v_y = \frac{dy}{dt} = \frac{d(t + 0.025\ t^3)}{dt}$$
$$= 1 + 0.025(3t^2) = 1 + 0.075\ t^2$$
$$\boldsymbol{v = (-0.5\ t)i + (1 + 0.075\ t^2)j}$$

At $t = 2s$ the velocity vector has components

$$v_x(2) = -0.5(2) = -\ 1\ \text{m/s}$$
$$v_y(2) = 1 + 0.075(2^2) = 1.3\ m/s$$

The magnitude of the instantaneous velocity (that is, the speed) at $t = 2s$:

$$v(2) = \sqrt{(-1)^2 + (1.3)^2} = 1.6\ m/s$$

The direction of the velocity vector , which is at an angle between 90^o and 180^o with respect to the positive x-axis.

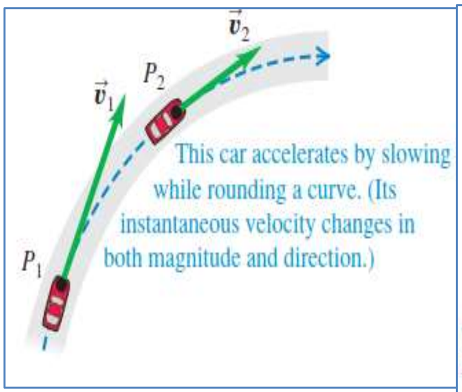
$arctan\frac{v_y}{v_x} = tan^{-1}\frac{v_y}{v_x} = tan^{-1}\frac{(1.3)}{(-1)} = -52^o$

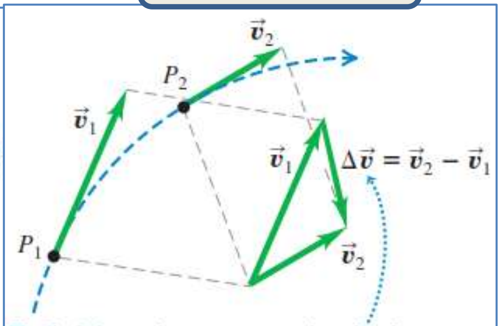
- The correct value of the angle is $180^o - 52^o = 128^o$

❖ The Acceleration Vector

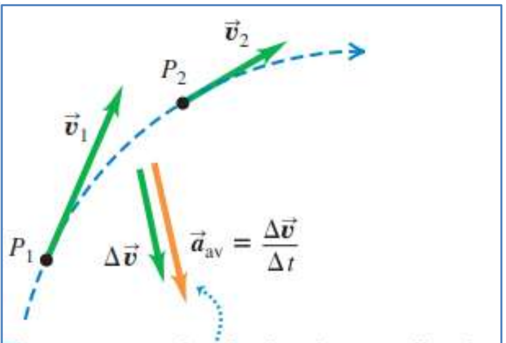
➢ The average acceleration:

$$\boxed{a_{av} = \frac{\Delta v}{\Delta t} = \frac{v_2 - v_1}{t_2 - t_1}}$$





To find the car's average acceleration between P_1 and P_2, we first find the change in velocity $\Delta\vec{v}$ by subtracting $\vec{v}_1$ from $\vec{v}_2$. (Notice that $\vec{v}_1 + \Delta\vec{v} = \vec{v}_2$.)



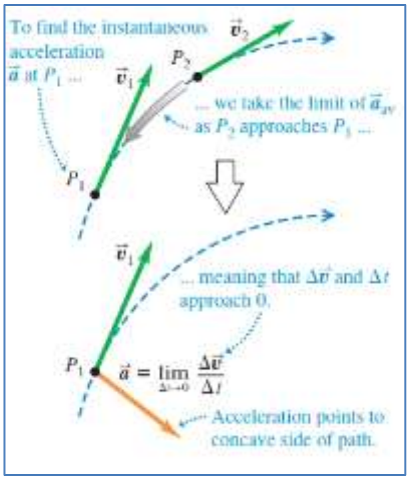
The average acceleration has the same direction as the change in velocity, $\Delta\vec{v}$.

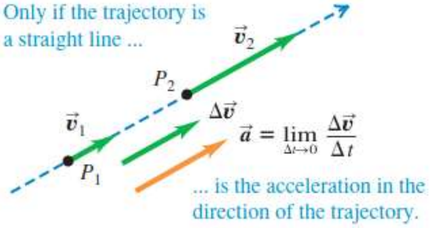
➢ the instantaneous acceleration:

The instantaneous acceleration is equal to the instantaneous rate of change of velocity with time

$$a = \lim_{\Delta t \to 0} \frac{\Delta v}{\Delta t} = \frac{dv}{dt}$$

- a points toward the inside of any turn that the particle is making.





- The acceleration is tangent to the path only if the particle moves in a straight line

Remember: When a particle is moving in a curved path, it always has **nonzero acceleration**, even when it moves with constant speed.

There is a nonzero acceleration whenever the velocity vector changes in any way, whether there is a change of speed, direction, or both.

- **In terms of unit vectors**
- $\boxed{a = a_x i + a_y j}$ $\boxed{a = \frac{dv}{dt} = \frac{dv_x}{dt} i + \frac{dv_y}{dt} j}$ $\boxed{a = \frac{d^2v}{dt^2} = \frac{d^2x}{dt^2} i + \frac{d^2y}{dt^2} j}$

➢ Calculating average and instantaneous acceleration

Example 2: (a) Find the components of the average acceleration for the interval $t = 0$ to $t = 2$ s. in example 1.

(b) Find the instantaneous acceleration at $t = 2\ s$.

(c) a general expression for the instantaneous acceleration vector $\boldsymbol{a}$. Express $a\ at\ t = 2s$ in component form and in terms of magnitude and direction.

Solution: from example 1

- $x = 2 - 0.25\ t^2$ ______________________ $y = t + 0.025\ t^3$
- $v_x = \frac{dx}{dt} = 0 - 0.25(2t) = -0.5\ t$ $\qquad v_y = \frac{dy}{dt} = 1 + 0.025(3t^2) = 1 + 0.075\ t^2$

At $t = 0$ the velocity vector has components

- $v_x(0) = -0.5(0) = 0$ $\qquad v_y(0) = 1 + 0.075(0^2) = 1\ m/s$

At $t = 2s$ the velocity vector has components

- $v_x(2) = -0.5(2) = -1 m/s$ $\qquad v_y(2) = 1 + 0.075(2^2) = 1.3 m/s$

(a) The components of average acceleration in the interval $t = 0\ to\ t = 2\ s$

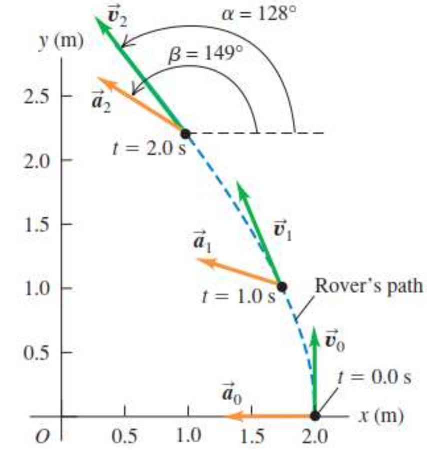
$$a_{av-x} = \frac{\Delta v_x}{dt} = \frac{-1-0}{2-0} = -0.5\ m/s^2$$

$$a_{av-y} = \frac{\Delta v_y}{dt} = \frac{1.3-1}{2-0} = -0.15\ m/s^2$$

(b)

$v_x = \frac{dx}{dt} = 0 - 0.25(2t) = -0.5\ t$ $\qquad v_y = \frac{dy}{dt} = 1 + 0.025(3t^2) = 1 + 0.075\ t^2$

$a_x = \frac{dv_x}{dt} = -0.5\ m/s^2$ $\qquad a_y = \frac{dv_y}{dt} = 0.075(2t) = 0.15\ t\ m/s^2$



At $t = 2s$ the acceleration vector has components

$a_x(2) = -0.5\ \ m/s^2$ $\qquad a_y(2) = 0.15(2) = 0.3 m/s^2$

The acceleration vector is $\quad \therefore a = -0.5\ i + 0.3\ j$

- Hence the instantaneous acceleration vector at time *t* is

$$a = a_x i + a_y j = -0.5\ i + 0.15\ t\ j$$

The magnitude of the instantaneous acceleration at $t = 2s$:

$$a(2) = \sqrt{(-0.5)^2 + (0.3)^2} = 0.58\ \ m/s^2$$

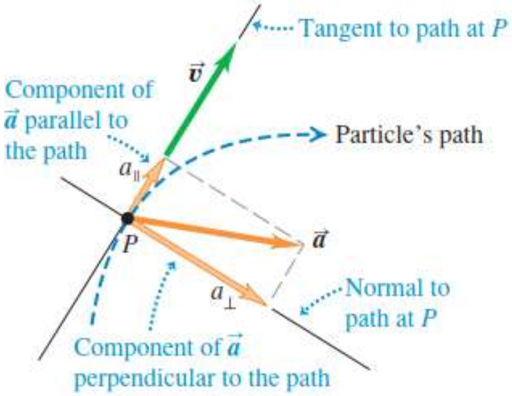
the direction of the acceleration vector , which is at an angle between 90^o and 180^o with respect to the positive x-axis.

$$tan^{-1}\frac{a_y}{a_x} = tan^{-1}\frac{(0.3)}{(-0.5)} = -31^o$$

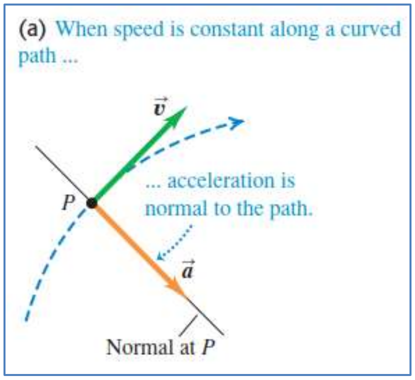
- The correct value of the angle is $\beta = 180^o - 31^o = 149^o$

❖ Parallel and Perpendicular Components of Acceleration

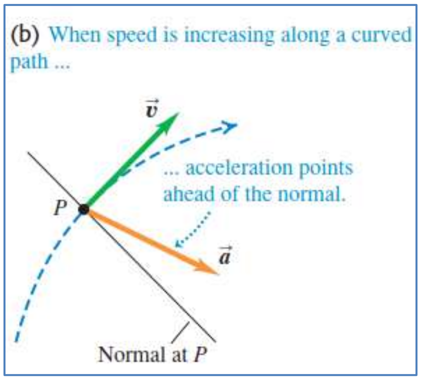
- The parallel component $a_{\parallel}$ tells us about changes in the particle's speed.
- The perpendicular component $a_{\perp}$ tells us about changes in the particle's direction of motion.



- The figure below shows a particle moving along a curved path for three different situations:
- Constant speed, increasing speed, and decreasing speed.

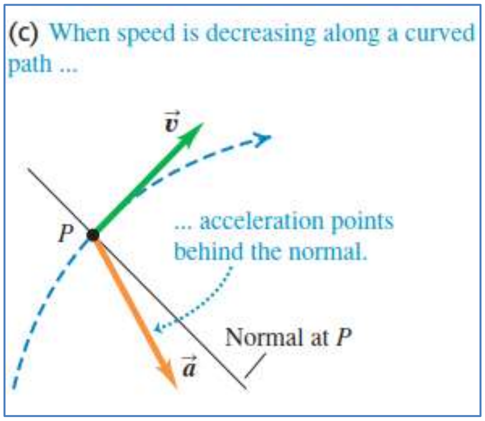

(a) When speed is constant along a curved path ...

- **(a) If the speed is constant, a is perpendicular to the path and to v and points toward the concave side of the path.**


(b) When speed is increasing along a curved path ...

(b) If the speed is increasing, there is still a perpendicular component of a but there is also a parallel component having the same direction as v. Then a points ahead of the normal to the path.

(c)


(c) When speed is decreasing along a curved path ...

If the speed is decreasing, the parallel component has the direction opposite to v and a points behind the normal to the path

➢ Calculating parallel and perpendicular components of acceleration

Example 3: find the parallel and perpendicular components of the acceleration at $t = 2s$ for the particle in example 1&2.

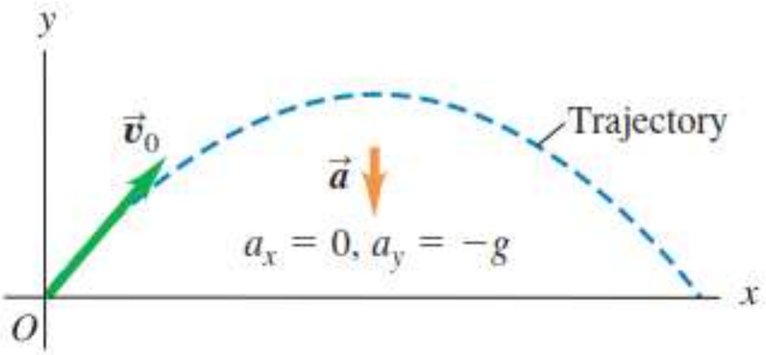
Solution: From Example2, at $t = 2s$ the particle has an accel. of magnitude 0.58 m/s^2 at an angle of 149^o w.r.t. positive x-axis. In Example1 we found that at this time the velocity vector is at an angle of 128^o w.r.t. the positive x-axis. The angle between $\boldsymbol{a}$ and $\boldsymbol{v}$ is therefore $149^o - 128^o = 21^o$. Hence the components of acceleration parallel and perpendicular to $\boldsymbol{v}$ are

- $a_{\|} = a\cos 21° = 0.58\cos 21° = 0.21\, m/s^2$
- $a_{\perp} = a\cos 21° = 0.58 sin\, 21° = 0.54\, m/s^2$

❖ Projectile Motion

A projectile is any object that is given an initial velocity and then follows a path determined entirely by the effects of gravitational acceleration and air resistance. we can analyze projectile motion as a combination of **horizontal** motion with **constant velocity** and **vertical** motion with **constant acceleration**. The components of are

$$\boxed{a_x = 0, \qquad a_y = -g}$$



➢ Projectile Motion: equations needed

- One dimensional, constant acceleration equations for **x** & **y** separately!

x part: **Acceleration** $\boldsymbol{a_x = 0}$!

y part: **Acceleration** $\boldsymbol{a_y = -g}$!

- The initial **x & y** components of the velocity: $\mathbf{v_{xi}}$ **&** $\mathbf{v_{yi}}$.

x motion: $\boldsymbol{v_{xf} = v_{xi}}$ = constant.

$$\boldsymbol{x_f = x_i + v_{xi}\, t \;; \quad (a_x = 0)}$$

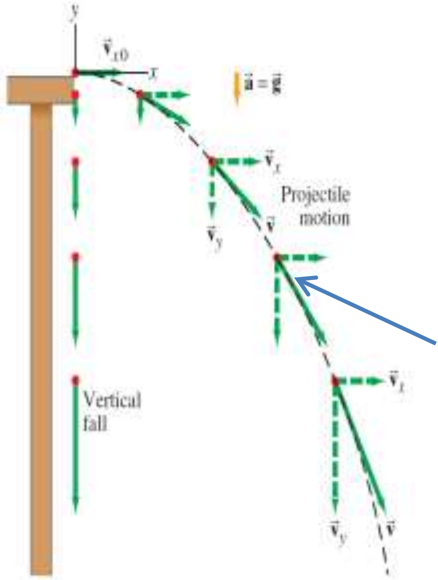
y motion: $\boldsymbol{v_{yf} = v_{yf} - gt}$

$$\boldsymbol{y_f = y_i + v_{yi}\, t - \frac{1}{2} g\, t^2}$$

$$\boldsymbol{(v_{yf})^2 = (v_{yi})^2 - 2g\,(y_f - y_i)}$$

- Developing the equations for certain situations

- The Ball Rolls Across the Table & Falls Off



Analyzing the horizontal & vertical motions separately. Take the motion down as positive. The initial velocity has an x component *ONLY*! That is $v_{yi} = 0$.

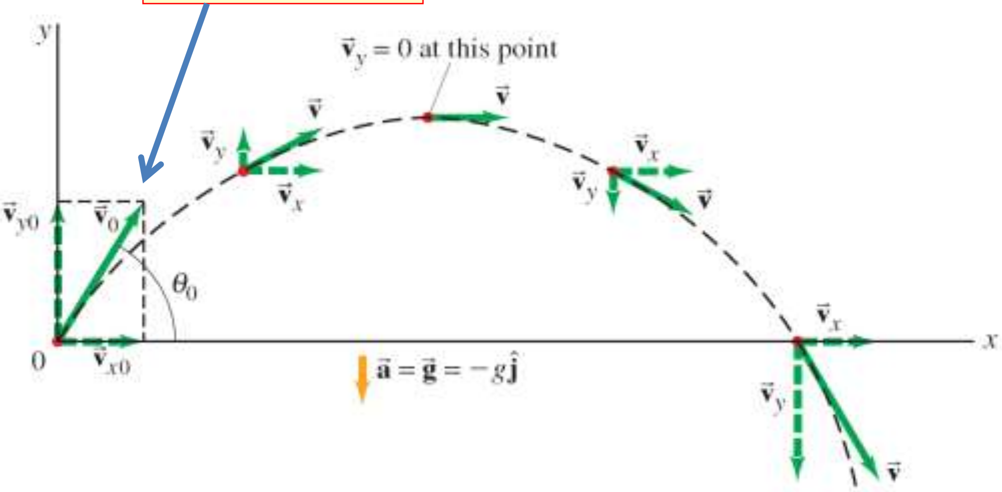
At any point, v has both x & y components. The kinematic equations tell us that, at time t,

$$v_x = v_{xi}$$
$$v_y = -gt$$
$$x_f = v_{xi}t$$
$$y_f = -(½)gt^2$$

- *General Case*: An object is launched at initial angle θ_0 with the horizontal. The analysis is similar to before, except the initial velocity has a vertical component $\boldsymbol{v_{y0}} \neq \mathbf{0}$.

Let upward be positive now!

$v_{x0} = v_0\cos\theta_0$
$v_{y0} = v_0\sin\theta_0$



The Parabolic shape of the path is real (neglecting air resistance!)

- Take ***y positive upward*** & origin at the point where it is shot: $\boldsymbol{x_i = y_i = 0}$

$\boldsymbol{v_{xi} = v_i\cos\theta_i}$, $\boldsymbol{v_{yi} = v_i\sin\theta_i}$

Horizontal motion:

NO ACCELERATION IN THE x DIRECTION $(a_x = 0)$

$$v_{xf} = v_{xi} \qquad x_f = v_{xi}\, t$$

Vertical motion:

$$v_{yf} = v_{yi} - gt$$

$$y_f = v_{yi}t - (½)gt2$$

$$(v_{yf})^2 = (v_{yi})^2 - 2gy_f$$

$$a_x = 0, \qquad a_y = -g = -9.8\, m/s^2$$

❖ Range & Maximum Height

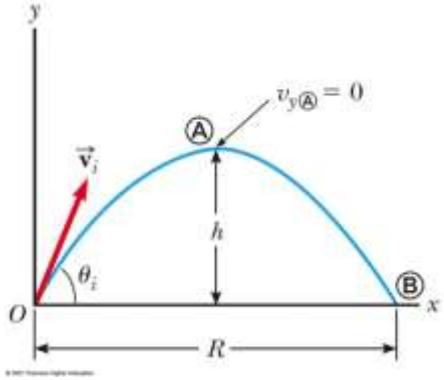
When analyzing projectile motion, **two characteristics are of special interest:**

- **The Range,**

R = the horizontal distance of the projectile

- **The Maximum Height**

h = the highest point.



- **The Maximum Height (h)** of a projectile is the height **at which the y component of velocity** $v_y = 0$.

It can be obtained from the kinematic equation:

$$(v_{yf})^2 = (v_{yi})^2 + 2g\,(y_f - y_i)$$

Assuming $y_i = 0$, setting $v_{yf} = 0$ & putting $y_f = h$:

$$0 = (v_{yi})^2 + 2gh$$

So: $h = [(v_{yi})^2/(2g)]$

Using $v_{yi} = v_i \sin\theta_i$ gives:

$$\boxed{h = \frac{v_i^2 \sin^2\theta_i}{2g}}$$

- **The Range**

The Range of a Projectile, $R = x_{max}$, is the **maximum horizontal distance from the starting point.**
It can be obtained from the kinematic equation

$$R = x_{max} = v_{xi}t_{max} \qquad (1)$$

With $t_{max} =$ *the time for it to hit the ground.*

The time to hit t_{max} can be obtained by setting $y_f = 0$ in the kinematic equation $y_f = v_{yi}\, t - (½)gt^2$.

This gives

$$t_{max} = (2v_{yi}/g) \qquad (2)$$

Putting **(2)** into **(1)** gives

$$R = (2v_{xi}v_{yi}/g) \qquad (3)$$

In **(3)**, using $v_{xi} = v_i cos\theta_i$ & $v_{yi} = v_i sin\theta_i$ along with the trig identity $2\ sin\theta_i cos\theta_i = sin(2\theta_i)$ gives the final result:

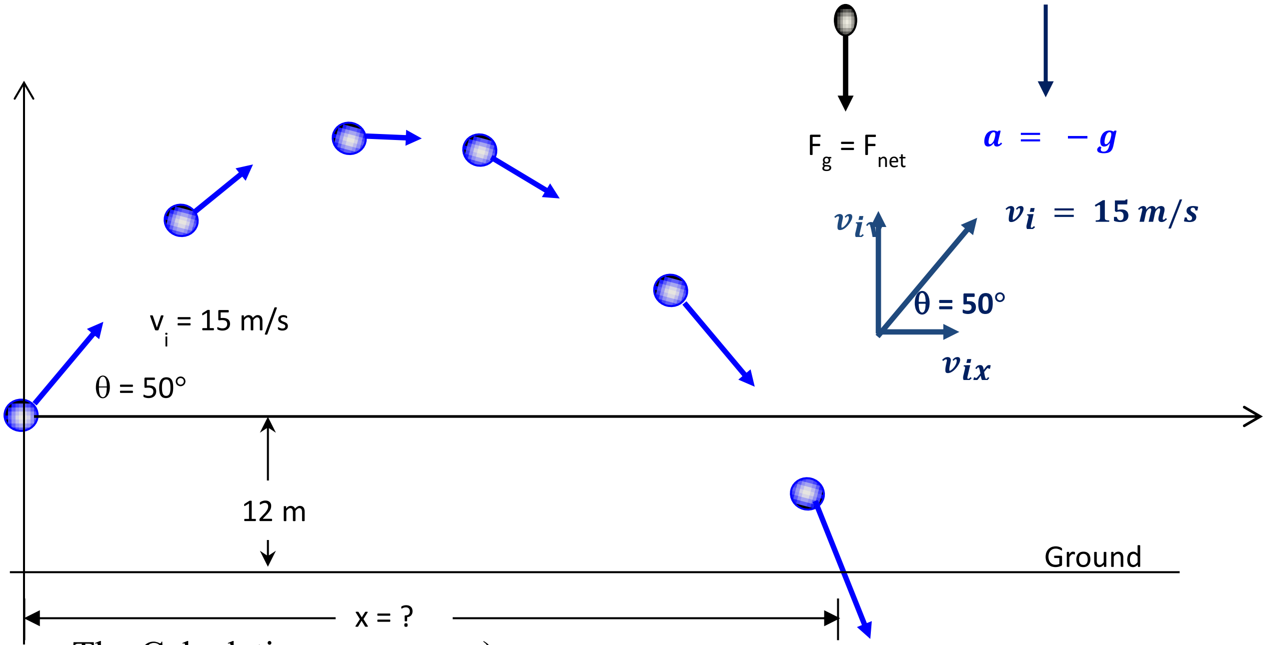
$$\boxed{R = \frac{v_i^2 \sin 2\theta_i}{g}}$$

- Using differential calculus, setting $(dR/d\theta i) = 0$ & solving for θi gives the result that

The maximum range **R** ***occurs at*** , $\theta i = 45^0$ ***independent of the initial velocity***.v_i

Example: A ball is thrown from the top of a building with a velocity of 15 m/s at an angle of 50 degrees above the horizontal.

- ■ What are the x and y components of the initial velocity?
- ■ What is the ball's maximum height?
- ■ If the height of the building is 12 m, where will the ball land?



➢ The Calculations of (y_{max})

◈ y-direction:

▪ Initial velocity: $v_{yi} = v_i\ sin\theta$

$$v_{yi} = \left(\frac{15\ m}{s}\right)(sin\ 50\ °) = 11.5\ m/s$$

Time when $v_{yf} = 0\frac{m}{s}$: $v_{yf} = v_{yi} - gt$

$t = v_{yi} / g$

$t = (11.5\ m/s)/(9.81\ m/s^2) = 1.17\ s$

Determine the maximum height: $y_{max} = y_i + v_{yi}t - ½\ gt^2$

- $y_{max} = 12\,m + \left(11.5\frac{m}{s}\right)(1.17\,s) - \frac{1}{2}\left(9.81\frac{m}{s^2}\right)(1.17\,s)^2 = 18.7\,m$

- The Calculations of (t)
  - Since the ball will accelerate due to gravity over the distance it is falling back to the ground, the time for this segment can be determined as follows
    - Time when ball hits the ground: $y_{max} = v_{yi}t - ½\,gt^2$
      - Since y_i can be set to zero as can v_{iy}

$$t = \sqrt{2 * ymax/g}$$

$$t = \sqrt{2(18.7\,m)/\,(9.81\,m/s^2)}$$

      - $t = 1.95\,s$
    - By adding the time it takes the ball to reach its maximum height to the time it takes to reach the ground will give you the total time.
      - $t_{total} = 1.17\,s + 1.95\,s = 3.12\,s$

- The Calculations of (x)
  - x-direction:
    - Initial velocity: $v_{ix} = v_i cos\theta$
      - $v_{ix} = (15\,m/s)(cos\,50°)$
      - $v_{ix} = 9.64\,m/s$
    - Determine the total distance: $x = v_{ix}t$
      - $x = (9.64\,m/s)(3.12\,s)$
      - $x = 30.1\,m$

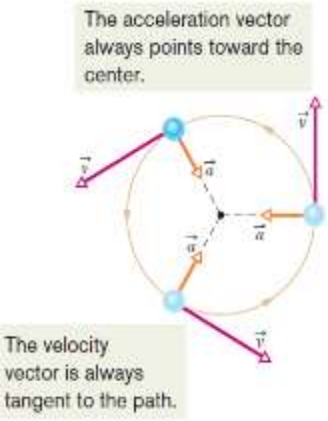
❖ Uniform Circular Motion

As the direction of the velocity of the particle changes, there is acceleration!!!

Centripetal Acceleration

$$a = \frac{v^2}{r} \qquad \text{(centripetal acceleration)},$$

Here v is the speed of the particle and r is the radius of the circle.


The acceleration vector always points toward the center.

The velocity vector is always tangent to the path.

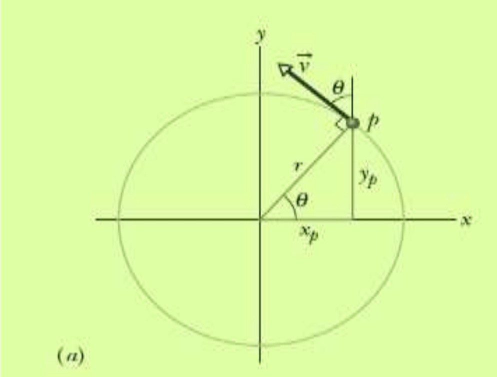
❖ Proof of $a = v^2/r$

$$v = v_x i + v_y j = (-vsin\theta)i + (vcos\theta)j$$

$$x_p = \mathrm{r}\cos\theta \qquad \therefore cos\theta = \frac{x_p}{r}$$

$$y_p = \mathrm{r}\sin\theta \qquad \therefore sin\theta = \frac{y_p}{r}$$

$$v = \left(-\frac{v\,.\,y_p}{r}\right)i + \left(\frac{v.\,x_p}{r}\right)j$$



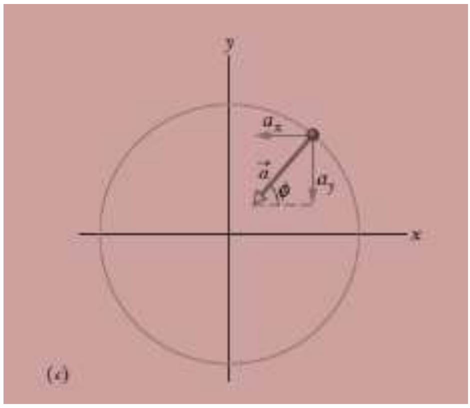
$$a = \frac{dv}{dt} = \left(-\frac{v}{r}\frac{dy_p}{dt}\right) i + \left(\frac{v}{r}\frac{dx_p}{dt}\right)$$

$v_x = -v\, sin\theta$

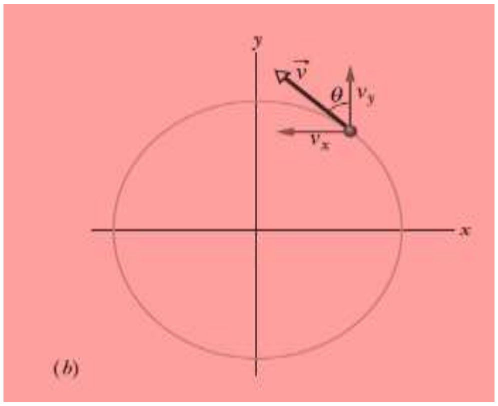
$v_y = -v\, \cos\theta$

Note: $\frac{dy_p}{dt} = v_y$ $\quad\quad$ $\frac{dx_p}{dt} = v_x$

$$a = \sqrt{{a_x}^2 + {a_y}^2} = \frac{v^2}{r}\sqrt{(cos\theta)^2 + (sin\theta)^2} = \frac{v^2}{r}\sqrt{1} = \frac{v^2}{r}$$



$$tan\varphi = \frac{a_y}{a_x} = \frac{-(v^2/r)sin\theta}{-(v^2/r)cos\theta}$$
$$= tan\theta$$



❖ Uniform Circular Motion

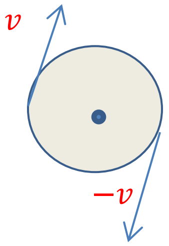
The magnitude of the acceleration in uniform circular motion can be expressed in terms of the period T of the motion, the time for one revolution (one complete trip around the circle).

In a time T the particle travels a distance equal to the circumference of the circle, so its speed is

$$v = \frac{2\pi R}{T}$$

$$\boldsymbol{a} = \frac{v^2}{R} = \frac{\boldsymbol{4\pi^2 R}}{\boldsymbol{T^2}}$$

Example 1: What is the magnitude of the acceleration, in g units, of a pilot whose aircraft enters a horizontal circular turn with a constant velocity $v_i = (400i + 500j) m/s$ and leaves the turn after $24s$ with a velocity $v_f = -v_i$ m/s ?



Solution: the time taken to complete one circle is

$$T = \frac{2\pi R}{v}$$

Therefore, $a = \frac{v^2}{R} = \frac{2\pi v}{T}$

$v = \sqrt{(400)^2 + (500)^2} = 640.3\, m/s$

Since $v_f = -v_i$ this means that the aircraft leaves the circle at the opposite side of the start point and must have completed half a circle in the given $24s$. Hence, the time for a complete circle is $48s$. So,

$$a = \frac{2\pi v}{T} = \frac{2\pi(640.31)}{48} = 83.8\, m/s^2$$

Example 2: A sport car has an acceleration of $0.96\ g$ which is the maximum centripetal acceleration the car can sustain without skidding out of the curved path. If the car is traveling at a constant velocity of $40\ m/s$ on level ground. What is the radius R of the tightest unbanked curve it can travel ?

Solution: $0.96\ g = 0.96 \times 9.8 = 9.4 m/s^2$

the radial acceleration is $a = \frac{v^2}{R}$, so

$$\boldsymbol{R = \frac{v^2}{a} = \frac{(40)^2}{9.4} = 170\ m}$$

- This is the minimum radius because $\boldsymbol{a}$ is the maximum centripetal acceleration

Lecture 5: Newton's Laws of Motion

Glossary

- ✓ **Inertia** is the tendency of an object to resist changes in its motion.
- ✓ A **force** is a push or pull on an object resulting from the object's interaction with another object. Forces only exist as a result of an interaction. Measured in newton (N).
- ✓ The **net force** is the vector sum of all the forces acting on an object.
- ✓ **Weight** is the Earth's gravitational force of attraction on a body and is directed towards the centre of the Earth.
- ✓ **Mass** refers to the quantity of matter in a body and is measured in kilograms (kg).

❖ 1st Law of Motion (Law of Inertia)

An object at rest will stay at rest, and an object in motion will stay in motion at constant velocity, unless acted upon by an unbalanced force.

Example: What would be your immediate resulting motion as a passenger on the bus if the bus performed the following manoeuvres…

a) A very quick start from rest?

b) A very sharp right-hand turn?

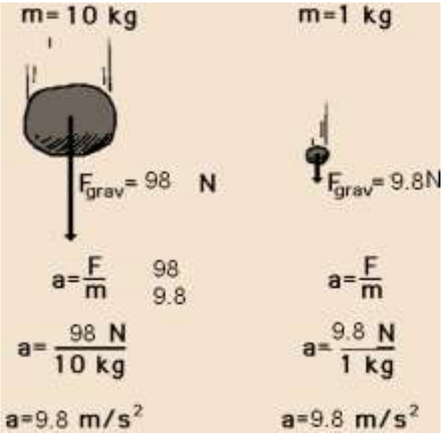
c) An emergency stop from a speed of 60 km/h?

❖ 2nd Law of Motion

The net force of an object is equal to the product of its mass and acceleration, or F=ma.

** The net force is the vector sum of all forces acting on the particle of mass m:*

$$F_{net} = \sum_i F_i = ma_i$$



F = ma
= 10 kg x 9.8
= 98 N

F = ma
= 1 kg x 9.8
= 9.8 N

- ➢ Newton's 2nd Law proves that different masses accelerate to the earth at the same rate, but with different forces.

Objects with different masses accelerate to the ground at the same rate. However, because of the 2nd Law we know that they **do NOT** hit the ground with the same force.

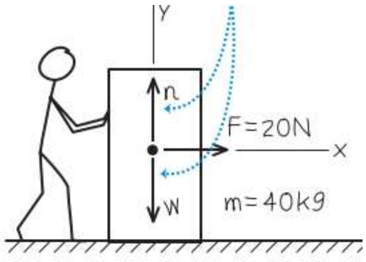
Mass and Force

Our results mean that for a given body, the ratio of the magnitude of the net force to the magnitude of the acceleration is constant, regardless of the magnitude of the net force. We call this ratio the ***inertial mas***s, or simply **the mass**, of the body and denote it by m. That is,

$$m = \frac{|\sum \vec{F}|}{a} \quad \text{or} \quad |\sum \vec{F}| = ma \quad \text{or} \quad a = \frac{|\sum \vec{F}|}{m}$$

$$1\ \mathrm{N} = 1\ \mathrm{kg \cdot m/s^2}$$

Example: A worker applies a constant horizontal force with magnitude 20 N to a box with mass 40 kg resting on a level floor with negligible friction. What is the acceleration of the box?



Solution: the horizontal net force $\sum F = ma = 20N$

$$\therefore a = \frac{\sum F}{m} = \frac{20\ N}{40\ kg} = \frac{20\ kg.m/s^2}{40\ kg} = 0.5\ m/s^2$$

Mass and wieght

To understand the relationship between **mass** and **weight**, note that a freely falling body has an acceleration of magnitude g. Newton’s second law tells us that a force must act to produce this acceleration. If a 1-kg body falls with an acceleration of 9.8 m/s^2 the required force has magnitude

$$F = ma = mg = 1\ kg \times 9.8 \frac{m}{s^2} = \ 9.8\ N = weight\ of\ the\ body\ (w)$$

$$\therefore w = mg$$

3rd Law of Motion

For every action, there is an equal and opposite reaction.

Example 1:

- A fish uses its fins to push water backwards. In turn, the water *reacts* by pushing the fish forwards, propelling the fish through the water.
- The size of the force on the water equals the size of the force on the fish; the direction of the force on the water (backwards) is opposite the direction of the force on the fish (forwards).

Example 2:

- Birds depend on Newton’s third law of motion. As the birds push down on the air with their wings, the air pushes their wings up and gives them lift.

❖ Applications of Newton’s laws of motion

➢ Using Newton's 1^{st} law: particles in equilibrium

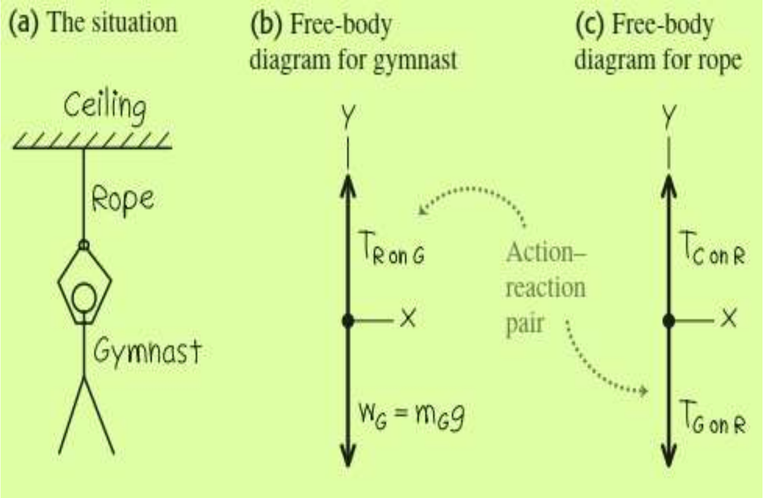
When a particle is in equilibrium, the net force acting on it—that is, the vector **sum of all the forces** acting on it—must **be zero**:

$$\sum \vec{F} = 0 \qquad \text{(particle in equilibrium, vector form)}$$

We most often use this equation in component form:

$$\sum F_x = 0 \qquad \sum F_y = 0 \qquad \text{(particle in equilibrium, component form)}$$

Example1: tension in a massless rope

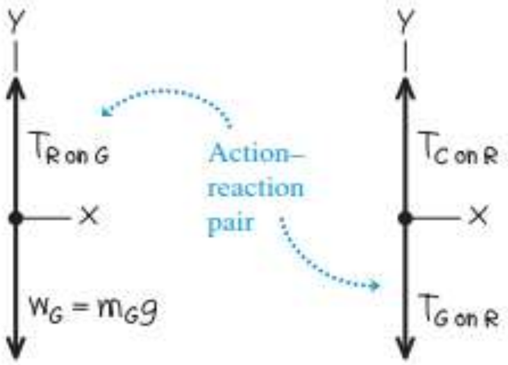


- A gymnast with mass $mG = 50.0\ kg$ suspends herself from the lower end of a hanging rope. The upper end of the rope is attached to the gymnasium ceiling.

a. What is the gymnast’s weight? What force (magnitude and direction does the rope exert on her?

b. What is the tension at the top of the rope? Assume that the mass of the rope itself is negligible.

Solution:



(a) $w = mg = 50 \times 9.8 \quad = 490\ N$

(b) $\sum F_y = T_R + (-w) = 0$

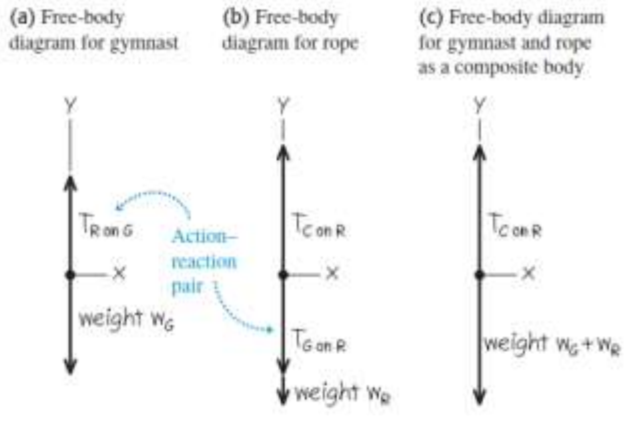
$$T_R = w = 490\ N$$

(c) $\sum F_y = T_{ceiling} + (-T_G) = 0$

$$T_{ceiling} = T_G = 490\ N$$

Example 2: tension in a rope with mass

Suppose that in example1, the weight of the rope is not negligible but is 120 N. find the tension at each end of the rope.


(a) Free-body diagram for gymnast (b) Free-body diagram for rope (c) Free-body diagram for gymnast and rope as a composite body

Solution:

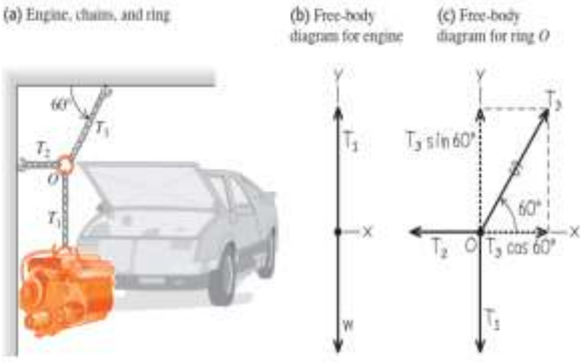
The equilibrium requires $\sum F_y = 0$

$$T_R + (-w_G - w_R) = 0$$

$$T_R = w_G + w_R = 490 + 120 = 610\ N$$

Example 3: two-dimensional equilibrium

A car engine with weight ***w*** hangs from a chain that is linked at ring ***O*** to two other chains, one fastened to the ceiling and the other to the wall. Find the tension in each of the three chains in terms of ***w,*** the weights of the ring and chains are negligible.


(a) Engine, chains, and ring (b) Free-body diagram for engine (c) Free-body diagram for ring *O*

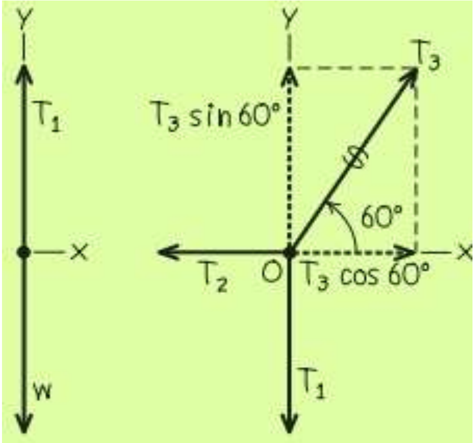
Solution: the forces acting on the engine are along the y-axis only, so Newton's 1st law

$$\sum F_y = T_1 - w = 0, \quad T_1 = w$$

The ring is in equilibrium, so the net force on the ring are zero:

$$\sum F_x = T_3 \cos 60^0 + (-T_2) = 0 \qquad (1)$$

$$\sum F_y = T_3 \sin 60^0 + (-T_1) = 0 \qquad (2)$$



$$T_3 = \frac{T_1}{sin60^0} = \frac{w}{sin60^0} = \frac{w}{\sqrt{3}/2} = 1.2\,w$$

$$T_2 = T_3 \cos 60^0 = \frac{w \cos 60^0}{sin60^0} = \frac{w/2}{\sqrt{3}/2} = \frac{w}{\sqrt{3}} = 0.58\,w$$

➢ Using Newton's 2nd law: dynamics of particles

We are now ready to discuss dynamics problems. In these problems, we apply Newton's second law to bodies on which the net force is not zero. These bodies are not in equilibrium and hence are accelerating. The **net force** on the body **is equal** to the **mass of the body times its acceleration**:

$$\sum \vec{F} = m\vec{a} \qquad \text{(Newton's second law, vector form)}$$

$$\sum F_x = ma_x \qquad \sum F_y = ma_y \qquad \text{(Newton's second law, component form)}$$

Example: Straight-line motion with a constant force

An iceboat is at rest on a frictionless horizontal surface. A wind is blowing along the direction of the runners so that 4.0 s after the iceboat is released, it is moving at 6 m/s. What constant horizontal force F_w does the wind exert on the iceboat? The combined mass of iceboat and rider 200 kg.

Solution: The **known quantities** are **the mass initial and final x-velocities** and the **elapsed ti**me. The three **unknown quantities** are **acceleration normal force n, and the horizontal force.** Hence we need three equations.

The first two equations are the x- and y-equations for Newton's second law. The force F_w is in the positive x-direction, while the forces n and w are in the positive and negative y-directions, respectively.

$$\sum F_x = F_W = ma_x$$
$$\sum F_y = n + (-mg) = 0 \quad \text{so} \quad n = mg$$

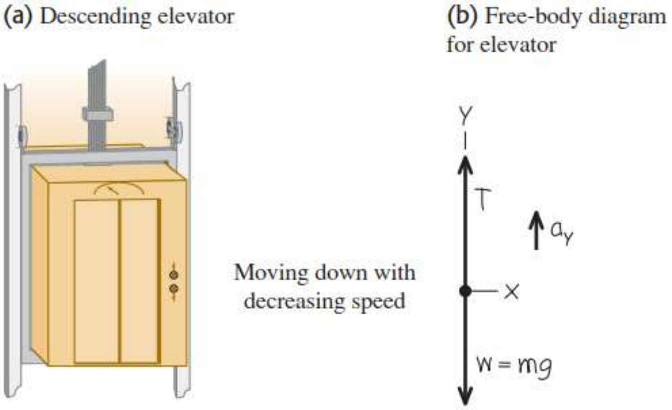
The third equation is the constant-acceleration relationship, $v_x = v_0 + a_x t$

$$\therefore a_x = \frac{v_x - v_0}{t} = \frac{6-0}{4} = 1.5\ m/s^2$$

$$\therefore F_w = ma_x = 200 \times 1.5 = 300\ N$$

Example 2: Tension in an elevator cable

An elevator and its load have a combined mass of 800 kg. The elevator is initially moving downward. It slows to a stop with constant acceleration in a distance of 25.0 m. What is the tension T in the supporting cable while the elevator is being brought to rest?



Solution: First let’s write out Newton’s second law. The tension force acts upward and the weight acts downward, so

$$\sum F_y = T + (-w) = ma_y$$

$$T = w + ma_y = mg + ma_y = m(g + a_y)$$

To determinea_y, use the constant-acceleration equation

$$v_y{}^2 = v_{0y}{}^2 + 2a_y(y - y_0)$$

$$a_y = \frac{v_y{}^2 - v_{0y}{}^2}{2(y - y_0)} = \frac{0 - (-10)^2}{2(0 - 25)} = 2m/s^2$$

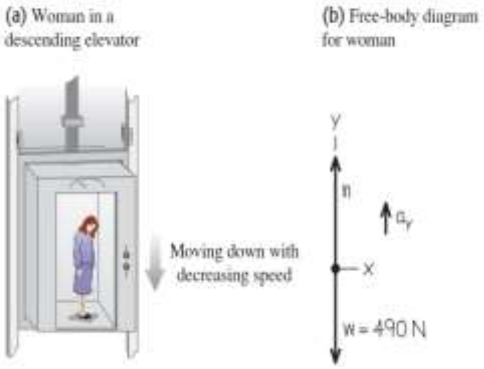
$$T = m(g + a_y) = 800(9.8 + 2) = 9440\ N$$

Example 3:Apparent weight in an accelerating elevator

A 50.kg woman stands on a bathroom scale while riding in the elevator in the previous Example.

(a) What is the reading on the scale?

(b) What would the woman feel if the elevator were accelerating downward



Solution: The scale reads the magnitude of the downward force exerted by the woman on the scale. By Newton’s third law, this equals the magnitude of the upward normal force exerted by the scale on the woman. Hence our target variable is the magnitude ***n*** of the normal force. We’ll find ***n*** by applying Newton’s second law to the woman. We already know her acceleration; it’s the same as the acceleration of the elevator, $a_y = 2\ m/s^2$

(a) Newton’s second law gives:

$$\sum F_y = n - mg = m\,a_y$$

$$n = mg + ma_y = m(g + a_y)$$

$$n = 50(9.8 + 2) = 590\ N$$

(b) if the elevator were accelerating downward, then $a_y = -2\ m/s^2$

$$n = m(g - a_y)$$

$$n = 50(9.8 - 2) = 390\ N$$

➢ Apparent weight and apparent weightlessness

- When a passenger with mass ***m*** rides in an elevator with y-acceleration a_y, a scale shows the passenger's apparent weight to be $n = m\cdot(g + a_y)$
- When the elevator is accelerating upward, a_y is positive and ***n*** is greater than the passenger's weight ***w*** = ***mg.*** when the elevator is accelerating downward, a_y is negative and ***n*** is less than the weight.
- When $a_y = g$, the elevator is in free fall, ***n*** = ***0*** and the passenger seems to be weightless.
- Similarly, an astronaut orbiting the earth in a spacecraft experiences apparent weightlessness.
- In each case, the person is not truly weightless because there is still gravitational force acting. But the person's sensation in this free-fall condition is exactly the same as though the person were in outer space with no gravitational force at all.

Example: Acceleration down a hill

A toboggan loaded with students (total weight w) slides down a snow-covered slope. The hill slopes at a constant angle, and the toboggan is so well waxed that there is virtually no friction. What is its acceleration?

Solution: The normal force *n* has only a y-component, but the weight has both x- and y-components.

The acceleration is purely in the +x-direction, so Newton's second law in component form then tells us that

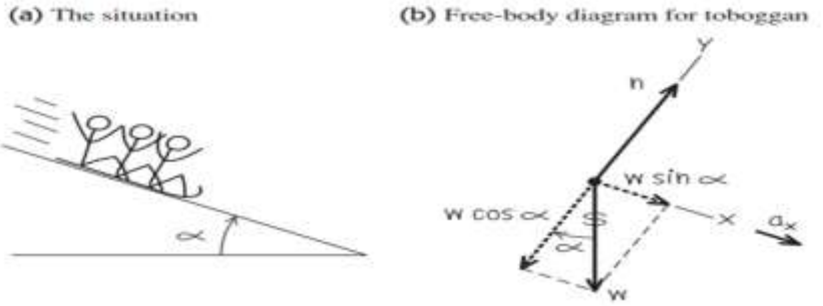
$$\sum F_x = w \sin\alpha = ma_x \qquad (1)$$

$$\sum f_y = n - w\cos\ \alpha = 0 \qquad (2)$$

Since $w = mg$, so from equation (1)

$a_x = mg \sin\alpha$

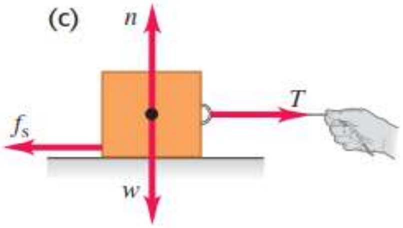
And from equation (2) $n = mg\cos\alpha$



Lecture 6: Frictional forces

- Friction force is a contact force; it usually exists when a body slides on a surface.
- The direction of friction force is always opposite to the direction of motion.
- There are two types of friction forces;
- ***kinetic frictions:*** friction that acts when a body slides over a surface f_k. The magnitude of the kinetic friction force can be represented by the equation:

$$f_k = \mu_k n$$



μ_k the coefficient of kinetic friction. μ_k depends on the types of surface. n is the force normal on the body.

- ***Static frictions:*** Friction forces may also act when there is no relative motion f_s.

- If you try to slide a box across the floor, the box may not move at all because the floor exerts an equal and opposite friction force on the box. $f_s \leq \mu_s n$

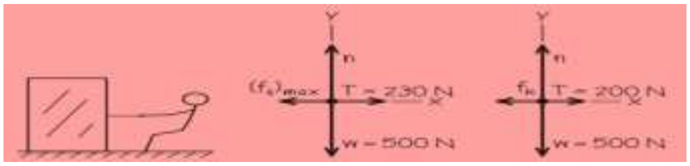
- The equality sign holds only when the applied force T has reached the critical value at which motion is about to start.

Example 1: Friction in a horizontal motion

You want to move a 500-N crate across a level floor. To start the crate moving, you have to pull with a 230-N horizontal force. Once the crate "breaks loose" and starts to move, you can keep it moving at constant velocity with only 200 N.

What are the coefficients of static and kinetic friction?

Solution:



Just before the crate moving

$\sum F_x = T + (-(f_s)_{max}) = 0$ so $(f_s)_{max} = T = 230\ N$

$\sum F_y = n + (-w) = 0$ so $n = w = 500\ N$

Thus, the coefficient of static friction is $\mu_s = \frac{f_s}{n} = \frac{230}{500} = 0.46$

After the crate starts to move

$\sum F_x = T + (-f_k) = 0$ so $f_k = T = 200\ N$

$\sum F_y = n + (-w) = 0$ so $n = w = 500\ N$

Thus, the coefficient of kinetic friction is $\mu_s = \frac{f_k}{n} = \frac{200}{500} = 0.40$

Example 2: Static friction can be less than the maximum

What is the friction force if the crate is at rest on the surface and a horizontal force of 50 N is applied to it?

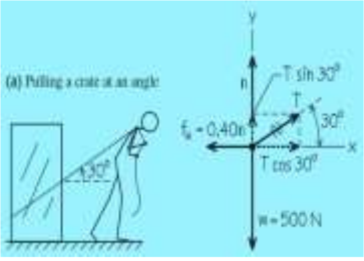
Solution: the maximum force of static friction is 230 N which is much larger than the applied force. Hence, the crate stay at rest and the net force acting on it is zero.

$\sum F_x = T + (-f_s) = 0$ so $f_s = T = 50\ N$

Example 3: Minimizing kinetic friction

Suppose you move the crate by pulling upward on the rope at an angle of 30^0above the horizontal. How hard must you pull to keep it moving with constant velocity? Assume that $\mu_k = 0.4$

Solution: The crate is in equilibrium because its velocity is constant, so we again apply Newton's first law. Since the crate is in motion, the floor exerts a kinetic friction force. $f_k = \mu_k n$



$$\sum F_x = T\cos 30° + (-f_k) = 0 \quad \text{so} \quad T\cos 30° = \mu_k n$$
$$\sum F_y = T\sin 30° + n + (-w) = 0 \quad \text{so} \quad n = w - T\sin 30°$$

$$T\cos 30° = \mu_k(w - T\sin 30°)$$

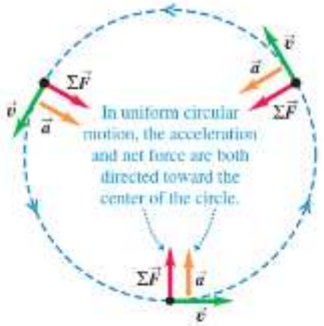
$$T = \frac{\mu_k w}{\cos 30° + \mu_k \sin 30°} = 188\ \text{N}$$

Therefore,

$$n = w - T\sin 30° = (500\ \text{N}) - (188\ \text{N})\sin 30° = 406\ \text{N}$$

❖ Dynamics of circular motion

When a particle moves in a circular path with constant speed, the particle's acceleration is always directed toward the center of the circle (perpendicular to the instantaneous velocity). The magnitude of the acceleration is constant and is given in terms of the speed and the radius R of the circle by $\boxed{a_{rad} = v^2/R}$, this acceleration is often called centripetal acceleration.



In uniform circular motion, the acceleration and net force are both directed toward the center of the circle.

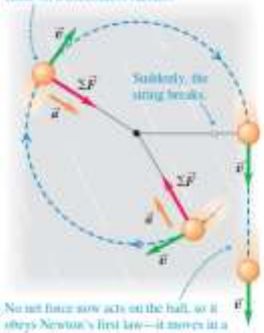
We can also express the centripetal acceleration in terms of the period T, the time for one revolution:

$$\boxed{T = \frac{2\pi R}{v}} \qquad \boxed{\therefore\ a_{rad} = \frac{4\pi^2 R}{T^2}}$$

Uniform circular motion is governed by Newton's second law.

- To make the particle accelerate toward the center of the circle, the net force F_{net} on the particle must always be directed toward the center.
- The magnitude of the acceleration is constant, so the magnitude of the net force must also be constant.
- If the inward net force stops acting, the particle flies off in a straight line tangent to the circle.

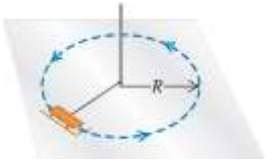
A ball attached to a string whirls in a circle on a frictionless surface.



No net force now acts on the ball, so it obeys Newton's first law—it moves in a straight line at constant velocity.

$$\boxed{F_{\text{net}} = ma_{\text{rad}} = m\frac{v^2}{R}}$$

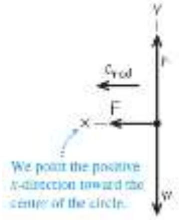
Example: Force in uniform circular motion



A sled with a mass of 25 kg rests on a horizontal sheet of essentially frictionless ice. It is attached by a 5m rope to a post set in the ice. Once given a push, the sled revolves uniformly in a circle around the post. If the sled makes five complete revolutions every minute, find the force F exerted on it by the rope.

Solution:

$$T = 60/5 = 12\ s$$



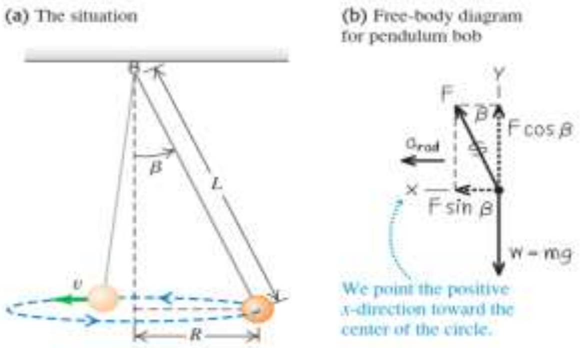
$$a_{rad} = \frac{4\pi^2 R}{T^2} = \frac{4\pi^2(5.00\ \text{m})}{(12.0\ \text{s})^2} = 1.37\ \text{m/s}^2$$

$F = m$

$= 34$

Example:A conical pendulum

An inventor designs a pendulum clock using a bob with mass m at the end of a thin wire of length L. Instead of swinging back and forth, the bob is to move in a horizontal circle with constant speed with the wire making a fixed angle β with the vertical direction. This is called a conical pendulum because the suspending wire traces out a cone. Find the tension F in the wire and the period T (the time for one revolution of the bob).


(a) The situation
(b) Free-body diagram for pendulum bob

Solution:

$\sum F_x = F\sin\beta = ma_{rad}$ (1)

$\sum F_y = F\cos\beta + (-mg) = 0$

$\sum F_y = F\cos\beta = (-mg)$ (2)

Divide equation (1) by (2) $\therefore a_{rad} = g\tan\beta$

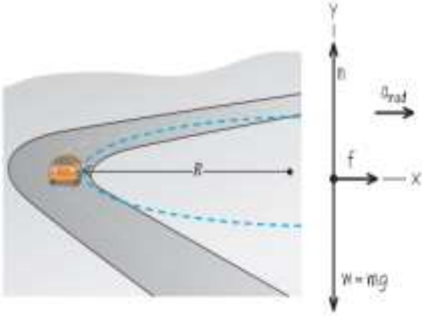
$$a_{rad} = \frac{4\pi^2 R}{T^2} \quad \longrightarrow \quad T = \sqrt{\frac{4\pi^2 R}{a_{rad}}} = \sqrt{\frac{4\pi^2 R}{g\tan\beta}} = 2\pi\sqrt{\frac{R}{g\ \tan\beta}}$$

$R = L\ \sin\beta$

$$\frac{\sin\beta}{\tan\beta} = \frac{\sin\beta}{\frac{\sin\beta}{\cos\beta}} = \cos\beta \qquad \therefore T = 2\pi\sqrt{\frac{L\ \cos\beta}{g}}$$

- ## Rounding a flat curve

Example : The sports car is rounding a flat, unbanked curve with radius R. If the coefficient of static friction between tires and road is μ_s. what is the maximum speed at which the driver can take the curve without sliding?



Solution: The acceleration toward the center of the circular path is

$a_{rad} = v^2/R.$

There is no vertical acceleration. Thus we have

$$\sum F_x = f = ma_{\text{rad}} = m\frac{v^2}{R}$$

$$\sum F_y = n + (-mg) = 0$$

The second equation shows that $n = mg$.

The first equation shows that the friction force *needed* to keep the car moving in its circular path increases with the car's speed. But the maximum friction force available is $f_{max} = \mu_s n = \mu_s mg$ and this determines the car's maximum speed.

Therefore, the first equation becomes $f_{max} = \mu_s mg = m\frac{v^2{}_{max}}{R}$

$$v_{max} = \sqrt{\mu_s gR}$$

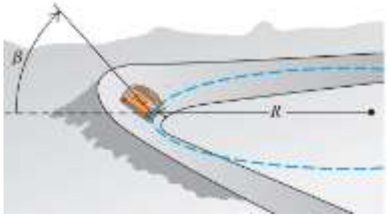
If $\mu_s = 0.96$ and $R = 230\,m$

$$v_{max} = \sqrt{0.96 \times 9.8 \times 230} = 47\frac{m}{s} = 170\frac{km}{h}$$

✓ This is the maximum speed for this radius.

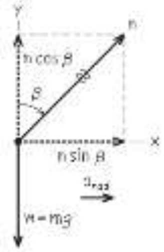
- **Rounding a banked curve**

  Example: At what angle β should the curve be banked so that a car moving at a chosen speed v can safely make the turn even with no friction?



Solution: The normal force $\boldsymbol{n}$ is perpendicular to the roadway and is at an angle β with the vertical. Thus it has a vertical component $n\,cos\,\beta$ and a horizontal component $n\,sin\,\beta$

The acceleration in the x-direction is the centripetal acceleration $a_{rad} = v^2/R$ there is no acceleration in the are:



$$\sum F_x = n\sin\beta = ma_{rad} \qquad (1)$$

$$\sum F_y = n\cos\beta + (-mg) = 0$$

$$n\cos\beta = mg \qquad (2)$$

Dividing (1) / (2) we get

$$tan\,\beta = \frac{a_{rad}}{g} = \frac{v^2}{gR} \quad so \quad \beta = \arctan\frac{v^2}{gR}$$

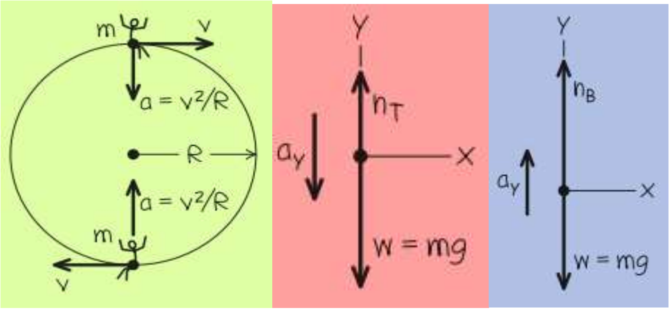
The bank angle depends on both the speed and radius. For

$$\mathrm{R} = 230m\ and\ \ v = 25\frac{m}{s}(= 88\ km/h)$$

$$\beta = \arctan\frac{25^2}{9.8 \times 230} = 15^0$$

- **Uniform circular motion in a vertical circle**

  Example: A passenger on a carnival Ferris wheel moves in a vertical circle of radius R with constant speed. The seat remains upright during the motion. Find expressions for the force the seat exerts on the passenger at the top of the circle and at the bottom.



Solution: At the top the acceleration has magnitude v^2/R but its vertical component is negative because its direction is downward.

- ✓ Top: $\sum F_y = n_T - mg = -\left(m\frac{v^2}{R}\right)$ *or* $n_T = mg(1 - \frac{v^2}{gR})$

At the bottom the acceleration is upward

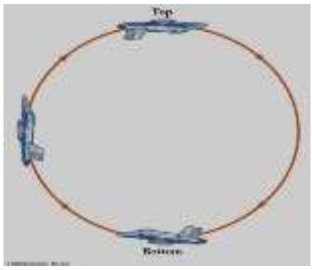
- ✓ Bottom: $\sum F_y = n_T - mg = +\left(m\frac{v^2}{R}\right)$ *or* $n_T = mg(1 + \frac{v^2}{gR})$

Example: A pilot of mass 70 kg in a jet aircraft executes a loop maneuver. The aircraft moves in a vertical circle of radius 2.7km at a constant speed of 225 m/s.

Determine the force exerted by the seat on the pilot at

(a) The bottom and,

(b) the top of the loop.



Solution: (a) the forces acting on the pilot are his downward weight mg and the upward normal force n_B exerted by the seat. Newton 2nd law for the upward direction gives

$$w = mg = 70 \times 9.8 = 686\ N$$

$$\sum F_y = n_B - mg = ma_{rad} = m\frac{v^2}{R}$$

$$n_B = mg + m\frac{v^2}{R} = \mathrm{mg}(1 + \frac{v^2}{gR})$$

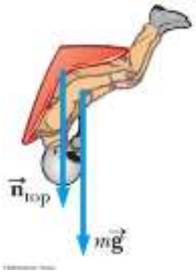
$$n_B = 70 \times 9.8\left(1 + \frac{225^2}{9.8 \times 2.7 \times 10^3}\right) = 1996\ N$$

$$1996/686 = 2.9$$

Therefore, the force exerted by the seat on the pilot at the bottom of the loop is 3 times his weight

(b) At the top of the loop both the gravitational force (his weight).

And the force exerted by the seat on the pilot act downward, the net force is



$$\sum F_y = n_T + mg = m\frac{v^2}{R}$$

$$n_T = m\frac{v^2}{R} - mg = mg(\frac{v^2}{gR} - 1)$$

$$n_T = 70 \times 9.8\left(\frac{225^2}{9.8 \times 2.7 \times 10^3} - 1\right)$$

$$= 625\ N$$

Therefore, the force exerted by the seat on the pilot at the top of the loop is less than his weight.

Lecture 7: Momentum, Impulse and Collisions

- Newton's second law for a particle in terms of the work–energy theorem

$$\sum F = ma$$

- Newton's Second Law in Terms of Momentum

$$\sum F = m\frac{dv}{dt} = \frac{d}{dt}(mv) \qquad (1)$$

the product of the particle's mass and

Velocity is called ***momentum*** **or** ***linear momentum*.**

$$p = mv \qquad (2)$$

The net force (vector sum of all forces) acting on a particle equals the time rate of change of momentum of the particle

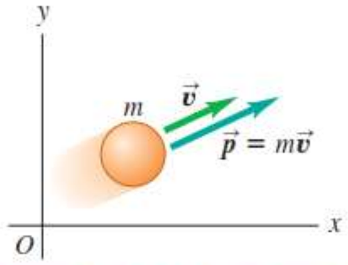
$$\sum F = \frac{dp}{dt}$$

Momentum and Impulse

Momentum is a vector quantity with the same direction as the particle's velocity.

If the particle has velocity components v_x, v_y and v_z then its momentum components p_x, p_y and p_z (which we also call the x-momentum, y-momentum, and z-momentum) are given by

$$p_x = mv_x \qquad p_y = mv_y \qquad p_z = mv_z \qquad (3)$$



Momentum $\vec{p}$ is a vector quantity; a particle's momentum has the same direction as its velocity $\vec{v}$.

Q) What is the value of the momentum of a 10 kg ball rolling down a bowling alley at a speed of 5 m/s?

$p = mv$ or $p = (10\,kg)(5\,m/s)$ or $p = 50\,kg\,m/s$

Example: Stopping Distance

A 2500 kg car brakes to slow from 25 m/s to 10 m/s in 6 s. What was the force of braking?

Solution:

$$\begin{aligned} F &= m\,\Delta v/\Delta t \\ &= (2500\,kg)(10\,m/s - 25m/s)/6s \\ F &= -6250\,N \end{aligned}$$

How far will it go in that time?

$$\begin{aligned} \Delta x &= ½(vi + vf)\Delta t \\ &= ½(25m/s + 10m/s)\,6s \\ \Delta x &= 105\,m \end{aligned}$$

The Impulse–Momentum Theorem

What is the fundamental difference between the momentum $(p = mv)$ and the kinetic energy $(\frac{1}{2}mv^2)$ of the particle.

- A purely mathematical answer is that momentum is a vector whose magnitude is proportional to speed, while kinetic energy is a scalar proportional to the speed squared.

- For the physical difference, introduce a quantity related to momentum called **Impulse.**
  - Consider a constant net force $\sum F$ acting on a particle during a time interval $(\Delta t = t_2 - t_1)$, then the product of the net force and the time interval is called The Impulse ***J.***

$$\boldsymbol{J = \sum F\Delta t = \sum F(t_2 - t_1)}$$

  - ❑ Impulse is a vector quantity; its direction is the same as the net force.
  - ❑ The SI unit of impulse is newton-second $(N.s)$, because $1N = kg.m/s^2$, so the unit of impulse is $kg.m/s$, the same unit as momentum.
  - ❑ If we now substitute the definition of momentum, Eq. (2), into Eq. (1), we get

$$\sum \boldsymbol{F} = \frac{d\boldsymbol{P}}{dt} \quad (Newton's\ second\ law\ in\ terms\ of\ momentum) \qquad (4)$$

If the net force $\sum F$ is constant, then is also $\frac{dp}{dt}$ constant. In that case, $\frac{dp}{dt}$ is equal to the total change in momentum $p_2 - p_1$ during the time interval $t_2 - t_1$ divided by the interval:

$$\sum F = \frac{p_2 - p_1}{t_2 - t_1}$$

Multiplying this equation by $t_2 - t_1$

$$\sum F\,(t_2 - t_1) = p_2 - p_1 = J \qquad (5)$$

The impulse–momentum theorem: The change in momentum of a particle during a time interval equals the impulse of the net force that acts on the particle during that interval.

$$J = p_2 - p_1 \qquad (6)$$

The impulse–momentum theorem also holds when forces are not constant.

$$\int_{t_1}^{t_2} \Sigma \vec{F}\, dt = \int_{t_1}^{t_2} \frac{d\vec{p}}{dt}\, dt = \int_{\vec{p}_1}^{\vec{p}_2} d\vec{p} = \vec{p}_2 - \vec{p}_1$$

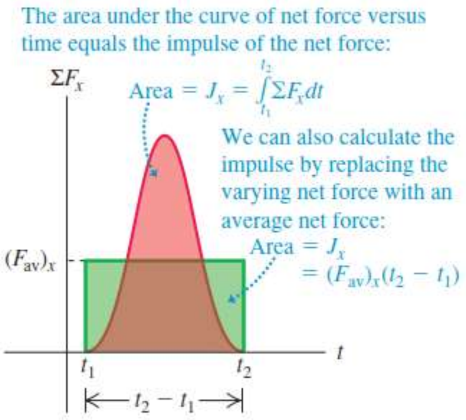
$$\vec{J} = \int_{t_1}^{t_2} \Sigma \vec{F}\, dt \qquad \text{(general definition of impulse)} \qquad (7)$$

With this definition, the impulse–momentum theorem is valid even when the net force varies with time.

We can define an average net force F_{av} such that even when the net force $\sum F$ is not constant the impulse is given by

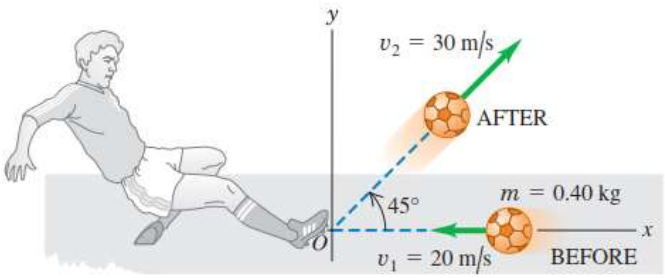
$$J = F_{av}(t_2 - t_1) \qquad (8)$$

When $\sum F$ is constant, $\sum F = F_{av}$ and Eq. (8) reduces to Eq. (5).



Example: A soccer ball has a mass of $0.40\ kg$. Initially it is moving to the left at but then it is kicked. After the kick it is moving at 45° upward and to the right with speed (see figure). Find the impulse of the net force and the average net force, assuming a collision time.

(a) Before-and-after diagram



Solution:

- **Velocity component before and after the kick:**

$v_{1x} = -20\ m/s \qquad v_{1y} = 0$

$v_{2x} = 30\ cos45 = 30 \times 0.707 = 21.2\ m/s$

$v_{2y} = 30\ sin45 = 30 \times 0.707 = 21.2\ m/s$

- **impulse–momentum theorem**

$$J_x = p_{2x} - p_{1x} = m(v_{2x} - v_{1x}) = 0.4(21.2 - (-20))$$
$$= 16.5 kg.m/s$$

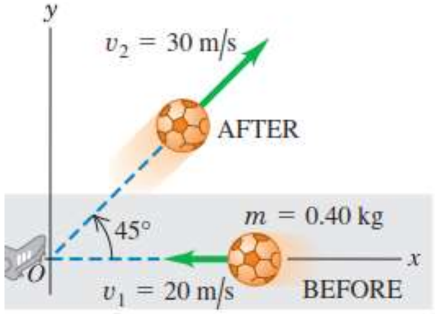
$$J_y = p_{2y} - p_{1y} = m(v_{2y} - v_{1y}) = 0.4(21.2 - 0) = 8.5\ kg.m/s$$

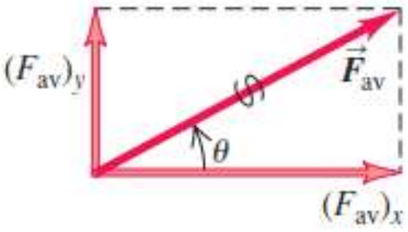
- **The average net force component**

$$(F_{av})_x = J_x/\Delta t = \frac{16.5}{0.01} = 1650\ N\ ,\ (F_{av})_y = J_y/\Delta t = \frac{8.5}{0.01} = 850N$$

- **The magnitude and direction of the average net force**

$$F_{av} = \sqrt{(1650)^2 + (850)^2} = 1.9 \times 10^3 N$$
$$\theta = arctan\frac{850}{1650} = 27^0$$





❖ Conservation of Momentum

- internal forces: the forces that the particles of the system exert on each other.
- external forces: Forces exerted on any part of the system by some object outside it.

The net force on particle A is $F_{B\ on\ A}$ and The net force on particle B is $F_{A\ on\ B}$. So the rates of change of the momenta of the two particles are

$$\vec{F}_{B\ on\ A} = \frac{d\vec{p}_A}{dt} \qquad \vec{F}_{A\ on\ B} = \frac{d\vec{p}_B}{dt}$$

The two forces $F_{B\ on\ A}$ and $F_{A\ on\ B}$ are always equal in magnitude and opposite in direction.

$$F_{B\ on\ A} = -F_{A\ on\ B} \longrightarrow F_{B\ on\ A} + F_{A\ on\ B} = 0$$

$$F_{B\ on\ A} + F_{A\ on\ B} = \frac{dp_A}{dt} + \frac{dp_B}{dt} = \frac{d(p_A + p_B)}{dt} = 0$$

The total momentum of the system of two particles as the vector sum of the momenta of the individual particles;

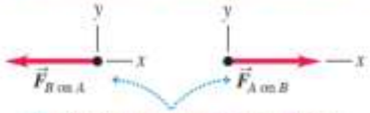
$$P = p_A + p_B + \cdots$$

Therefore $F_{B\ on\ A} + F_{A\ on\ B} = \frac{dP}{dtt} = 0$

The time rate of change of the total momentum is zero.

8.8 Two astronauts push each other as they float freely in the zero-gravity environment of space.

No external forces act on the two-astronaut system, so its total momentum is conserved.

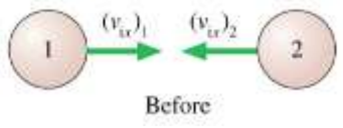


The forces the astronauts exert on each other form an action–reaction pair.

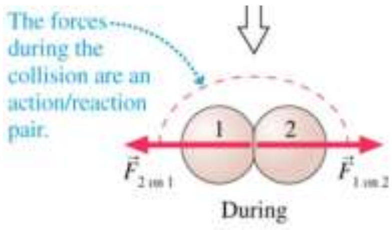
❖ principle of conservation of momentum

- **If the vector sum of the external forces on a system is zero, the total momentum of the system is constant.**
- The total momentum before equals the total momentum after, if there are no external forces.

$$m_1 v_{1i} + m_2 v_{2i} = m_1 v_{1f} + m_2 v_{2f}$$



Consider 2 objects speeding toward each other. When they collide......



Due to Newton's 3rd Law the FORCE they exert on each other are EQUAL and OPPOSITE

The TIMES of impact are also equal.

$$t_1 = t_2$$

Therefore, the IMPULSES of the 2 objects colliding are also EQUAL

$$(Ft)_2 = (Ft)_1$$

$$J_2 = -J_1$$

If the Impulses are equal then the MOMENTUMS are also equal!

$$p_2 = -p_1$$

$$\boldsymbol{m_2 \Delta v_2 = -m_1 \Delta v_1}$$

$$m_2 v_{fx,2} - m_2 v_{ix,2} = -\left(m_1 v_{fx,1} - m_1 v_{ix,1}\right)$$

$$m_1 v_{ix,1} + m_2 v_{ix,2} = m_1 v_{fx,1} + m_2 v_{fx,2}$$

$$p_{ix,1} + p_{ix,2} = p_{fx,1} + p_{fx,2}$$

$$p_{ix,total} = p_{fx,total}$$

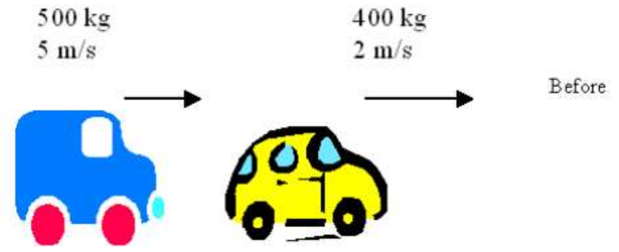
The Law of Conservation of Momentum: "In the absence of an external force (gravity, friction), the total momentum before the collision is equal to the total momentum after the collision."

❖ Elastic and Inelastic Collisions

- **Energy is not conserved in a perfectly inelastic collision.**
  - **If the objects bounce apart instead of sticking together, the collision is either elastic or partially inelastic.**
    - **An elastic collision is one in which no energy is lost.**
    - **A partially inelastic collision is one in which some energy is lost, but the objects do not stick together.**
    - **The greatest portion of energy is lost in the perfectly inelastic collision, when the objects stick.**

- **A ball bouncing off a floor or wall with no decrease in the magnitude of its velocity is an elastic collision.**
  - **The kinetic energy does not decrease.**
  - **No energy has been lost.**
- **A ball sticking to the wall is a perfectly inelastic collision.**
  - **The velocity of the ball after the collision is zero.**
  - **Its kinetic energy is then zero.**
  - **All of the kinetic energy has been lost.**
- **Most collisions involve some energy loss, even if the objects do not stick, because the collisions are not perfectly elastic.**
  - **Heat is generated, the objects may be deformed, and sound waves are created.**
  - **These would be partially inelastic collisions.**

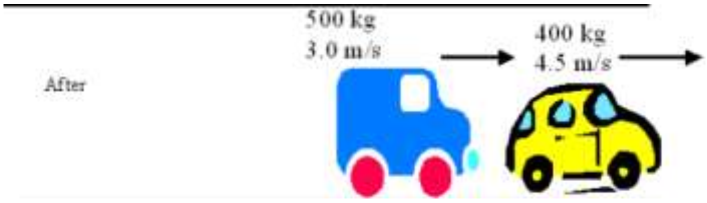
Example



$$p_{o(truck)} = mv_o = (500)(5) = 2500 kg * m/s$$

$$p_{o(car)} = (400)(2) = 800 kg * m/s$$

$$p_{o(total)} = 3300 kg * m/s$$



$$p_{truck} = 500 * 3 = 1500 kg * m/s$$

$$p_{car} = 400 * 4.5 = 1800 kg * m/s$$

$$p_{total} = 3300 kg * m/s$$

- **Types of collisions**
  - Sometimes objects stick together or blow apart. In this case, momentum is ALWAYS conserved.

$$\sum p_{before} = \sum p_{after}$$

$$m_{total} v_{o(total)} = m_1 v_1 + m_2 v_2 \longrightarrow$$ When 1 object breaks into 2 objects

$$m_1 v_{01} + m_2 v_{02} = m_{total} v_{total} \longrightarrow$$ When 2 objects collide and stick together

$$m_1 v_{01} + m_2 v_{02} = m_1 v_1 + m_2 v_2 \longrightarrow$$ When 2 objects collide and DON'T stick

Elastic Collision = Kinetic Energy is Conserved

Inelastic Collision = Kinetic Energy is NOT Conserved

Example: A bird perched on an 8.00 cm tall swing has a mass of 52.0 g, and the base of the swing has a mass of 153 g.
Assume that the swing and bird are originally at rest and that the bird takes off horizontally at 2.00 m/s. If the base can swing freely (without friction) around the pivot, how high will the base of the swing rise above its original level?
How many objects due to have BEFORE the action? 1.
How many objects do you have AFTER the action? 2.

$$P_{Before} = P_{After}$$

$$m_T v_{oT} = m_1 v_1 + m_2 v_2$$

$$(0.052 + 0.153) \times 0 = 0.153 \times \mathrm{v}_{1(swing)} + 0.052 \times 2$$

$$v_{swing} = -0.680 \text{ m/s}$$

Elastic Collision = Kinetic Energy is Conserved

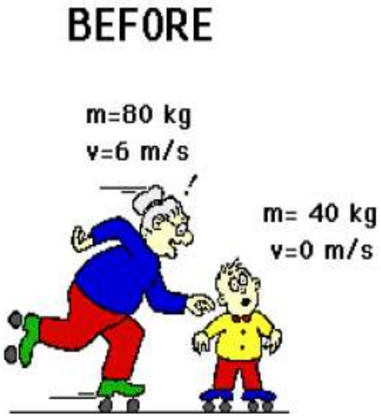
$$E_B = E_A$$

$$K_{o(swing)} = U_{swing}$$

$$\frac{1}{2} m v_o^2 = mgh$$

$$\frac{v_o^2}{2g} = h = \frac{(0.68)^2}{19.6} = 0.024 \text{ m}$$

Example: Granny (m=80 kg) whizzes around the rink with a velocity of 6 m/s. She suddenly collides with her grandson (m=40 kg) who is at rest directly in her path. Rather than knock him over, she picks him up and continues in motion without "braking." Determine the velocity of Granny and grandson.
How many objects do I have before the collision? 2
How many objects do I have after the collision? 1

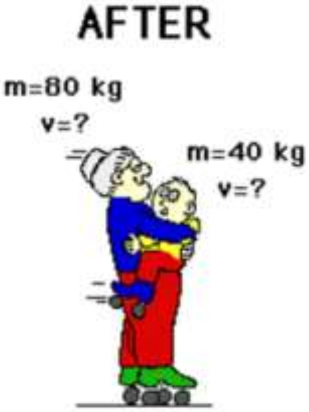


$$P_{Before} = P_{After}$$

$$m_1 v_{o1} + m_2 v_{o2} = m_T v_T$$

$$(80)(6) + (40)(0) = 120 v_T$$

$$v_T = 4 \text{ m/s}$$

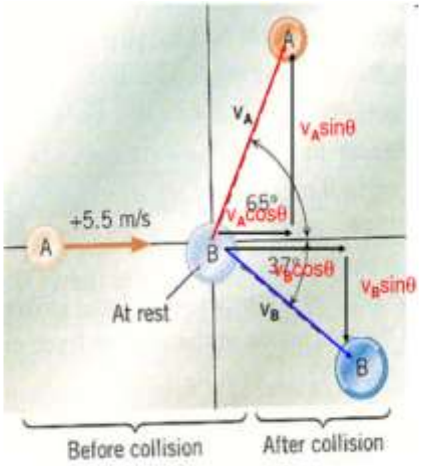


Inelastic Collision = Kinetic Energy is NOT Conserved

❖ Collisions in 2 Dimensions

Example: a collision between two pucks on an air hockey table. Puck A has a mass of 0.025kg and is moving along the x-axis with a velocity of +5.5 m/s. It makes a collision with puck B, which has a mass of 0.050kg and is initially at rest. The collision is NOT head on. After the collision, the two pucks fly apart with angles shown in the drawing.

Calculate the speeds of the pucks after the collision.



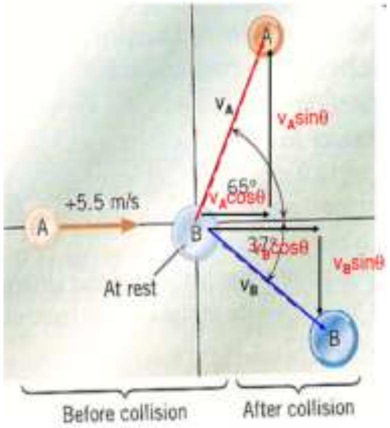
Solution:

$$\sum p_{ox} = \sum p_x$$

$$m_A v_{oxA} + m_B v_{oxB} = m_A v_{xA} + m_B v_{xB}$$

$$(0.025)(5.5) + 0 = (.025)(v_A \cos 65) + (.050)(v_B \cos 37)$$

$$0.1375 = 0.0106 v_A + 0.040 v_B$$



$$\sum p_{oy} = \sum p_y$$

$$0 = m_A v_{yA} + m_B v_{yB}$$

$$0 = (0.025)(v_A sin65) + (0.05)(-v_B sin37)$$

$$0.03 v_B = 0.0227 v_A$$

$$v_B = 0.757 v_A$$

$$0.1375 = 0.0106 v_A + 0.040 v_B$$

$$0.1375 = 0.0106 v_A + 0.04 \times 0.757\ v_A$$

$$0.1375 = 0.0106 v_A + (0.04)(0.757 v_A)$$

$$0.1375 = 0.0106 v_A + 0.0303 v_A$$

$$0.1375 = 0.04088 v_A$$

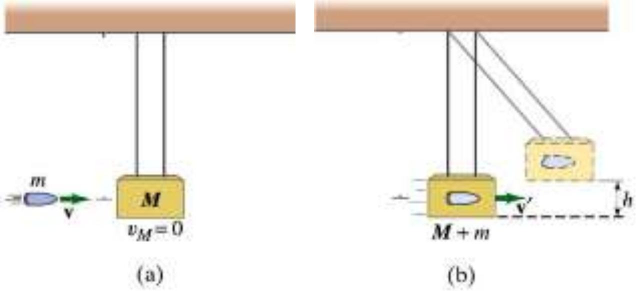
$$\therefore v_A = 3.363$$

Therefore $v_B = 0.757 \times 3.363 = 2.546$

Example:

A bullet of mass m and velocity Vo plows into a block of wood with mass M which is part of a pendulum.

1. How high, h, does the block of wood go?
2. is the collision elastic or inelastic?



(a) (b)

Solution:

$$\sum p_{before} = \sum p_{after}$$

$$\boxed{\begin{array}{l} x:\quad m\text{v} + 0 = (M+m)\text{v}' \\ y:\quad 0+0 = 0+0 \end{array} \Rightarrow \text{v}' = \frac{m\text{v}}{(M+m)}}$$

$$E_{bottom} = E_{top}$$

$$\frac{1}{2}(M+m)(\text{v}')^2 + 0 = 0 + (M+m)gh$$

$$\Rightarrow h = \frac{1}{2g}(\text{v}')^2 = \frac{m^2\text{v}^2}{2g(m+M)^2}$$

Lecture 8: Center of Mass

What is the “Center of Mass?”

This is a special point in space where “it is as if the object could be replaced by all the mass at that one little point”.

Examples where this is useful:

- We can model the earth moving around the sun as a single point at “the center of the earth”
- There is only one point on a stick that you can put your finger under and hold it up

How do you calculate CM?

1. Select an origin
2. Look at each “piece of mass” and figure out how much mass it has and how far it is (vector displacement) from the origin.
3. Multiply mass by position
4. Add them all up and divide out by the sum of the masses

- The center of mass is a displacement vector *“relative to some origin”*

The math:

$$X_{CM\ for\ 2\ particles} = \frac{m_1x_1 + m_2x_2}{m_1 + m_2}$$

$$X_{CM\ for\ 3\ particles} = \frac{m_1x_1 + m_2x_2 + m_3x_3}{m_1 + m_2 + m_3}$$

So the x coordinate of the center of mass of *n* particles is defined as

$$\boldsymbol{X_{CM} =, \frac{\sum_i m_ix_i}{\sum_i m_i} = \frac{\sum_i m_ix_i}{M} = \frac{1}{M}\sum_i m_ix_i\,,\ \ i = 1, 2, 3, \ldots n}$$

- The center of mass can be located in three dimensions by its position vector r_{CM}

$$\boldsymbol{r_{CM} = \frac{1}{M}\sum_i^n m_ir_i}$$

- $r_i = x_i\boldsymbol{i} + y_i\boldsymbol{j} + z_i\boldsymbol{k}$
- $i = 1,2,3, \ldots .n$

❖ The Center of Mass of Three Particles

Example: A system consists of three particles located as shown. Find the center of mass of the system. The masses of the particles are $m_1 = m_2 = 1\ kg$ and $m_3 = 2\ kg$.

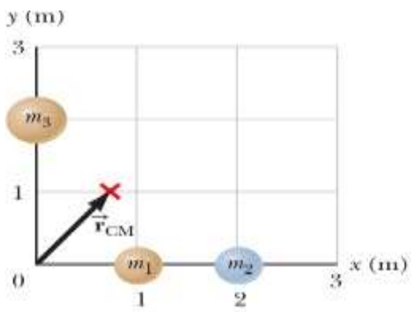
Solution:

$$x_{CM} = \frac{1}{M}\sum_i m_i x_i = \frac{m_1x_1 + m_2x_2 + m_3x_3}{m_1 + m_2 + m_3}$$

$$= \frac{(1.0\text{ kg})(1.0\text{ m}) + (1.0\text{ kg})(2.0\text{ m}) + (2.0\text{ kg})(0)}{1.0\text{ kg} + 1.0\text{ kg} + 2.0\text{ kg}} = \frac{3.0\text{ kg}\cdot\text{m}}{4.0\text{ kg}} = 0.75\text{ m}$$

$$y_{CM} = \frac{1}{M}\sum_i m_i y_i = \frac{m_1y_1 + m_2y_2 + m_3y_3}{m_1 + m_2 + m_3}$$

$$= \frac{(1.0\text{ kg})(0) + (1.0\text{ kg})(0) + (2.0\text{ kg})(2.0\text{ m})}{4.0\text{ kg}} = \frac{4.0\text{ kg}\cdot\text{m}}{4.0\text{ kg}} = 1.0\text{ m}$$



$$\boldsymbol{r}_{CM} = x_{CM}\boldsymbol{i} + y_{CM}\boldsymbol{j} = 0.75\boldsymbol{i} + \boldsymbol{j}$$

❖ Motion of the Center of Mass

- **What happens to the center of mass when the particles move?**

We know that $v_x = \frac{dx}{dt}$

$$v_{CM-x} = \frac{m_1v_{1x}+m_2v_{2x}+m_3v_{3x}+\cdots}{m_1+m_2+m_3}$$

$$v_{CM-y} = \frac{m_1v_{1y}+m_2v_{2y}+m_3v_{3y}+\cdots}{m_1+m_2+m_3}$$

$$\boldsymbol{v}_{CM} = \frac{d\boldsymbol{r}}{dt} = \frac{m_1\boldsymbol{v_1}+m_2\boldsymbol{v_2}+m_3\boldsymbol{v_3}+\cdots}{M},$$

Where $M = m_1 + m_2 + m_3$

$$\therefore M\boldsymbol{v}_{CM} = m_1\boldsymbol{v_1} + m_2\boldsymbol{v_2} + m_3\boldsymbol{v_3} + \cdots = \boldsymbol{P}$$

- Thus we have proved that *the total momentum is equal to the total mass times the velocity of the center of mass.*

For a system of particles on which the net external force is zero, so that the total momentum $\boldsymbol{P}$ is constant, the velocity of the center of mass $\boldsymbol{v}_{CM}$ is also constant.

❖ External Forces and Center-of-Mass Motion

- If the net external force on a system of particles is not zero, then total momentum is not conserved and the velocity of the center of mass changes.

$$\boldsymbol{a}_{CM} = \frac{d\boldsymbol{v}_{CM}}{dt}$$

- the acceleration of the center of mass

$$M\boldsymbol{a}_{CM} = m_1\boldsymbol{a}_1 + m_2\boldsymbol{a}_2 + m_3\boldsymbol{a}_3 + \cdots$$

- we can classify each force as *external* or *internal*. The sum of all forces on all the particles is then,

$$\sum \boldsymbol{F} = \sum \boldsymbol{F}_{ext} + \sum \boldsymbol{F}_{int} = M\boldsymbol{a}_{CM}$$

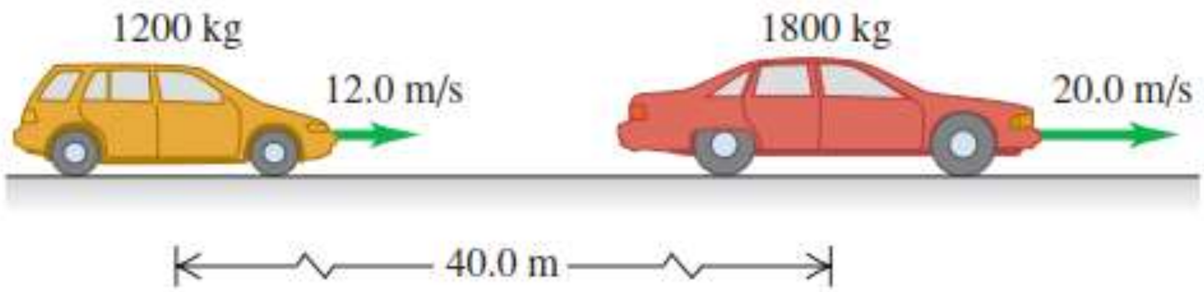
- Due to Newton's third law, the internal forces all cancel in pairs, and $\sum \boldsymbol{F}_{int} = 0$

$$\sum \boldsymbol{F}_{ext} = M\boldsymbol{a}_{CM} = \frac{d\boldsymbol{P}}{dt}$$

Example: A 1200-kg station wagon is moving along a straight highway at 12 m/s. Another car, with mass 1800 kg and speed 20 m/s has its center of mass 40.0 m ahead of the center of mass of the station wagon.

(a) Find the position of the center of mass of the system consisting of the two automobiles.

(b) Find the magnitude of the total momentum of the system from the given data.
(c) Find the speed of the center of mass of the system.
(d) Find the total momentum of the system, using the speed of the center of mass. Compare your result with that of part (b).



Solution:

$m_A = 1200\text{ kg},\ \ m_B = 1800\text{ kg}.\ \ M = m_A + m_B = 3000\text{ kg}.$

Let +x be to the right and let the origin be at the center of mass of the station wagon.

$$\textbf{(a)}\ \ x_{\text{cm}} = \frac{m_A x_A + m_B x_B}{m_A + m_B} = \frac{0 + (1800\text{ kg})(40.0\text{ m})}{1200\text{ kg} + 1800\text{ kg}} = 24.0\text{ m}.$$

The center of mass is between the two cars, 24.0 m to the right of the station wagon and 16.0 m behind the lead car.

$$\textbf{(b)}\ \ P_x = m_A v_{A,x} + m_B v_{B,x} = (1200\text{ kg})(12.0\text{ m/s}) + (1800\text{ kg})(20.0\text{ m/s}) = 5.04\times10^4\text{ kg}\cdot\text{m/s}.$$

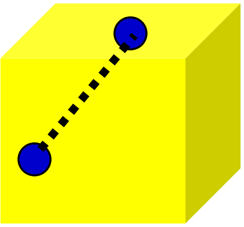
$$\textbf{(c)}\ \ v_{\text{cm},x} = \frac{m_A v_{A,x} + m_B v_{B,x}}{m_A + m_B} = \frac{(1200\text{ kg})(12.0\text{ m/s}) + (1800\text{ kg})(20.0\text{ m/s})}{1200\text{ kg} + 1800\text{ kg}} = 16.8\text{ m/s}.$$

$$\textbf{(d)}\ \ P_x = Mv_{\text{cm}-x} = (3000\text{ kg})(16.8\text{ m/s}) = 5.04\times10^4\text{ kg}\cdot\text{m/s},\ \text{the same as in part (b).}$$

The total momentum can be calculated either as the vector sum of the momenta of the individual objects in the system, or as the total mass of the system times the velocity of the center of mass.

Lecture 9: Rotation of rigid body

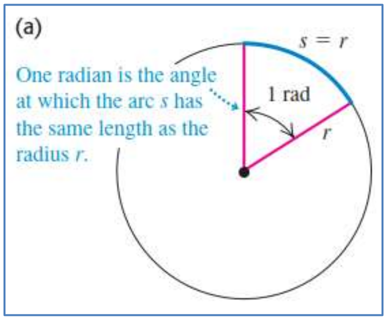
1. Rigid body



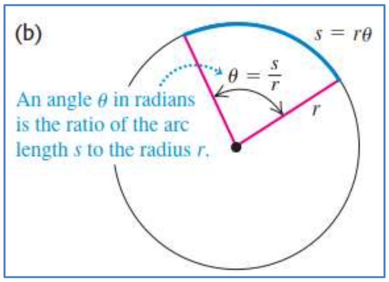
The body that has a perfectly definite and unchanged shape and size no matter how much external force acts on it.
The distance between two points in a rigid body maintains constant forever.

2. Motion forms of rigid body

Translation motion: Motion of a body in which every point of the body moves parallel to and the same distance as every other point of the body.



A radian (1 rad) is the angle subtended at the center of a circle by an arc with a length equal to the radius of the circle.



An angle θ is subtended by an arc of length s on a circle of radius r. The value of θ (in radians) is equal to s divided by r:

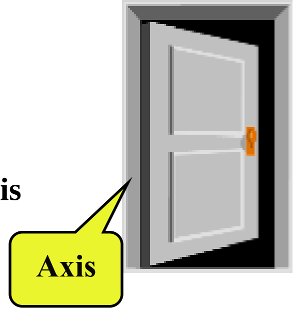
$$\theta = \frac{s}{r} \qquad s = r\theta$$

The circumference of a circle (that is, the arc length all the way around the circle) is 2π times the radius, so there are 2π (about 6.283) radians in one complete revolution(360^0). Therefore

$$1\ rad = \frac{360^0}{2\pi} = 57.3^0$$

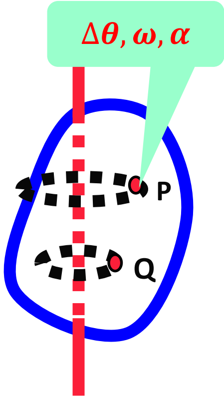
$$1\ \text{rev/s} = 2\pi\ \text{rad/s} \quad \text{and} \quad 1\ \text{rev/min} = 1\ \text{rpm} = \frac{2\pi}{60}\ \text{rad/s}$$

- **Rotation**
  - **Rotating around axis**
    - **Fixed axis**
    - **Moving axis**
  - **Rotating around a point**



Superposition of several rotations around axis

Rotation of a rigid body around a fixed axis



Every point of the rigid body moves in a circle

They have the same angular displacement, angular speed & angular acceleration.
Select arbitrary point P. P's rotation can represent the rotation of the rigid body.

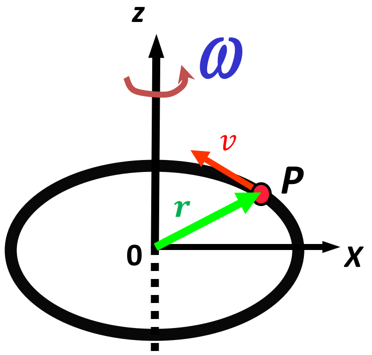
Angular position θ
Angular displacement $\Delta\theta$
Axis
Angular speed

$$\omega = \frac{d\theta}{dt}$$

Angular acceleration $\alpha = \frac{d\omega}{dt} = \frac{d^2\theta}{dt^2}$

Relation between angular and linear quantities

$$\begin{tikzcd} \theta \arrow[d] & \Delta\theta \arrow[d] & \omega \arrow[d] & \alpha \arrow[d] \arrow[dr] & \\ \mathbf{r} & \Delta s & v & a_t & a_{rad} \end{tikzcd}$$



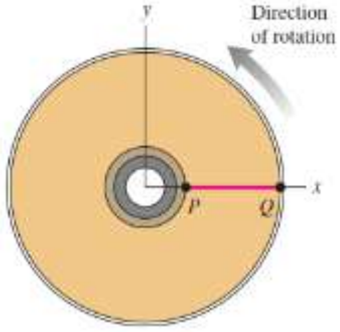
$$s = r\theta \qquad v = r\omega$$

$$a_t = r\alpha \qquad a_{rad} = \omega^2 r$$

Rotation with constant angular acceleration

Straight-Line Motion with Constant Linear Accel.	Fixed-Axis Rotation with Constant Angular Accel.
$a_x = constant$	$\alpha_z = constant$
$v_x = v_{0x} + a_x t$	$\omega_z = \omega_{0z} + \alpha_z t$
$x = x_0 + v_{0x} t + \frac{1}{2} a_x t^2$	$\theta = \theta_0 + \omega_{0z} t + \frac{1}{2} \alpha_z t^2$
${v_x}^2 = {v_{0x}}^2 + 2a_x(x - x_0)$	${\omega_x}^2 = {\omega_{0z}}^2 + 2\alpha_z(\theta - \theta_0)$
$x - x_0 = \frac{1}{2}(v_x + v_{0x})t$	$\theta - \theta_0 = \frac{1}{2}(\omega_z + \omega_{0z})$

Example: A disc is slowing to a stop. The disc's angular velocity at $t = 0$ is $27.5\ rad/s$ and its angular acceleration is a constant $-10\ rad/s^2$. A line *PQ* on the disc's surface lies along the $+x - axis$ at $t = 0$.
What is the disc's angular velocity at $= 0.3\ s$?
What angle does the line *PQ* make with the x-axis at this time?



Solution:

(a) $\omega_z = \omega_{0z} + \alpha_z t = 27.5 + (-10)(0.3) = 24.5\ rad/s$

(b) $\theta = \theta_0 + \omega_{0z} t + \frac{1}{2}\alpha_z t^2 = 0 + 27.5(0.3) + \frac{1}{2}(-10)(0.3)^2$

$= 7.8\ rad$

$Remember \ldots .1\ rad = \frac{360^0}{2\pi} = 57.3^0$

$\theta = 7.8 \times 57.3^0 = 446.94 - 360 = 86.94 \approx 87^0$

Hence, the line PQ makes an angle of 87^0 with $+x - axis$

ROTATIONAL ENERGY

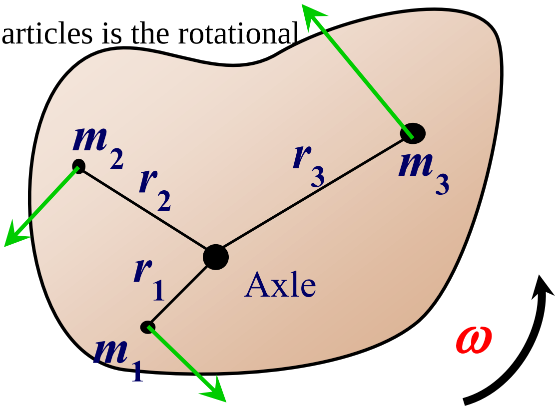
Each particle in a rigid rotating body has kinetic energy.
The sum of all the individual kinetic energies of each of the particles is the rotational kinetic energy of the body

$K_{rot} = \frac{1}{2} m_1 v_1^2 + \frac{1}{2} m_2 v_2^2 + \ldots$

$\therefore K_{rot} = \frac{1}{2} m_1 r_1^2 \omega^2 + \frac{1}{2} m_2 r_2^2 \omega^2 + \ldots$

$\therefore K_{rot} = \frac{1}{2} (\Sigma m_i r_i^2) \omega^2$

$\therefore K_{rot} = \frac{1}{2} I \omega^2$



Moment of inertia

$I = \sum_i m_i r_i^2$ --the rotational inertia (Moment of inertia) of the rigid body about the axis

It describes the rotating inertia of a rigid body about an axis.

Calculation of moment of inertia

The magnitude of moment of inertia depends on the total mass and mass distribution of body, the location and orientation of the axis.

① Discrete particles $I = \sum_i (m_i r_i^2)$

② Continuous distribution of mass $I = \int r^2 dm$

the mass distribution over a line $dm = \lambda\, dl$ where λ – the mass per unit length

the mass distribution over a surface $dm = \sigma\, ds$ where σ – the mass per unit area

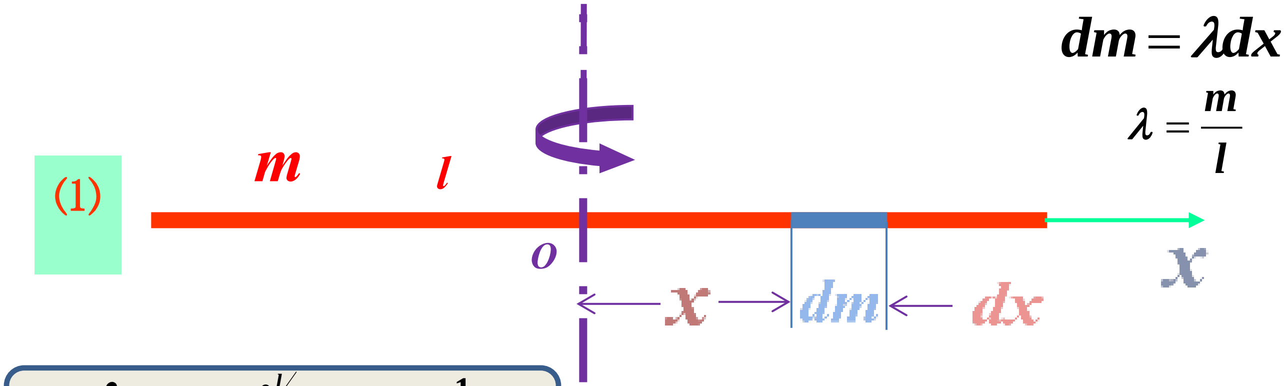
the mass distribution over a volume $dm = \rho\, dv$ where ρ – the mass per unit volume

Example: A slender rod has mass m, length l. Find its I about some axis follow as

(1) the axis through O and perpendicular with the rod.

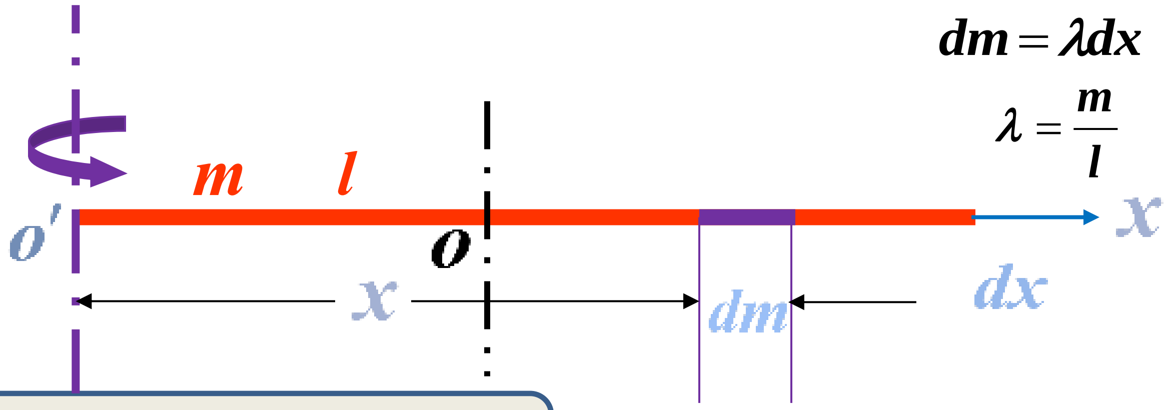
(2) the axis through an end of the rod and perpendicular with it.

(3) the axis at arbitrary distance h from O



$$\boxed{I_0 = \int r^2 dm = \int_{-l/2}^{l/2} \lambda x^2 dx = \frac{1}{12} m l^2}$$

(2)

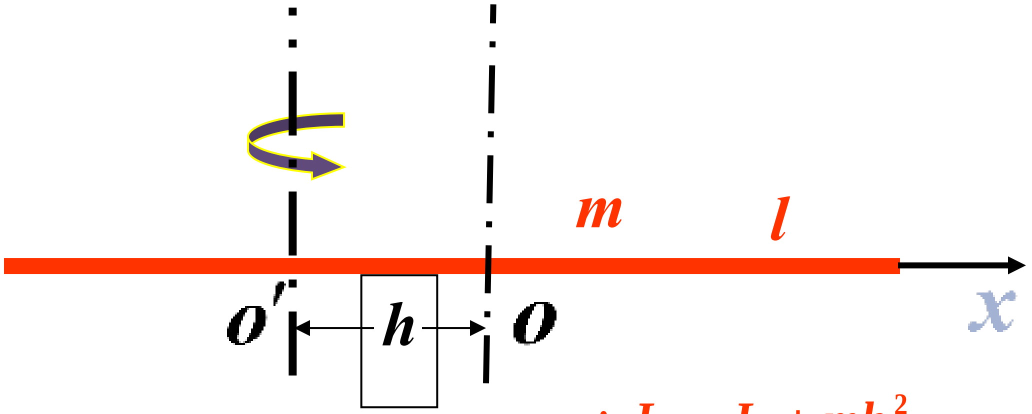


$$dm = \lambda dx$$

$$\lambda = \frac{m}{l}$$

$$I = \int r^2 dm = \int_0^l \lambda x^2 dx = \frac{1}{3} m l^2$$

(3)

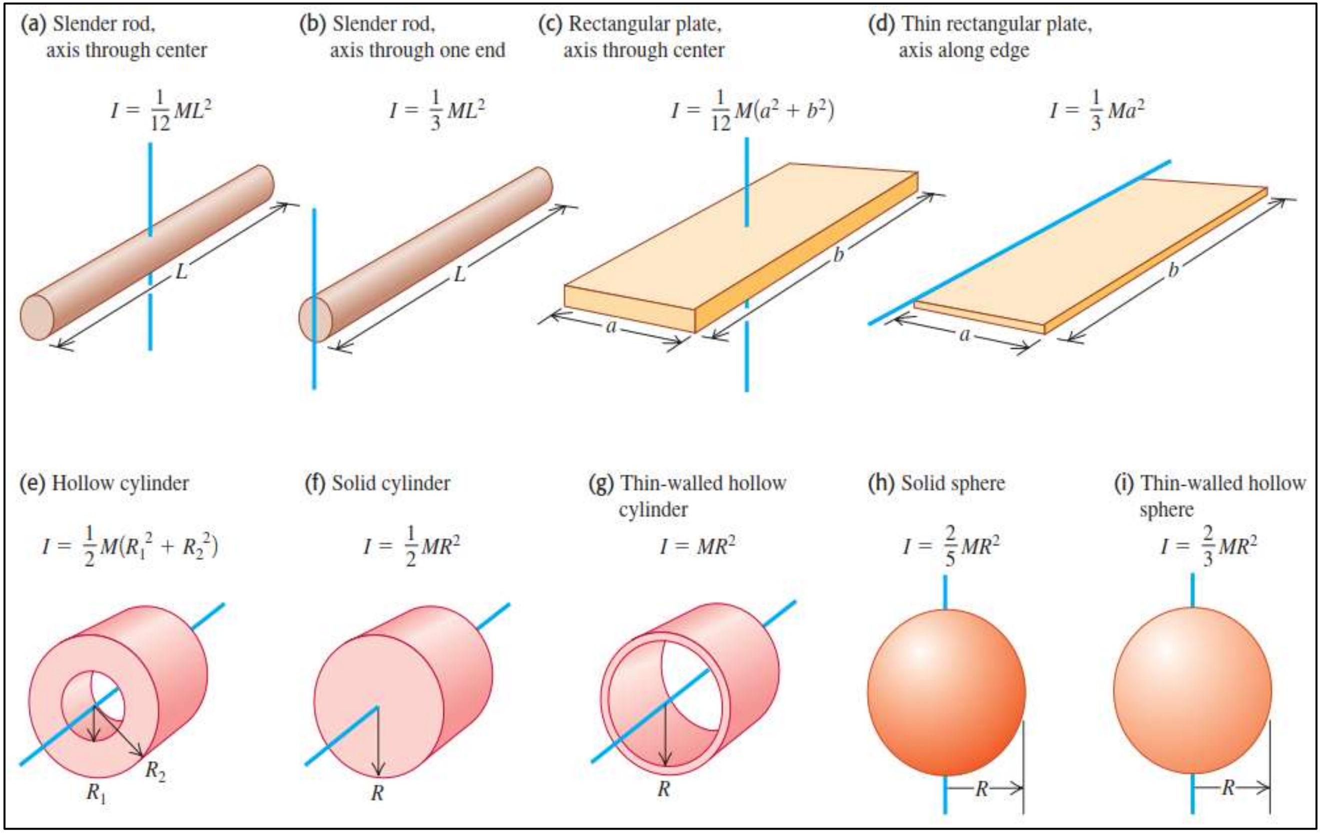


$$I_h = \int_{-(\frac{l}{2}-h)}^{\frac{l}{2}+h} \lambda x^2 dx = \frac{1}{12} m l^2 + m h^2$$

$$\therefore I_h = I_0 + m h^2$$

--Parallel-axis theorem

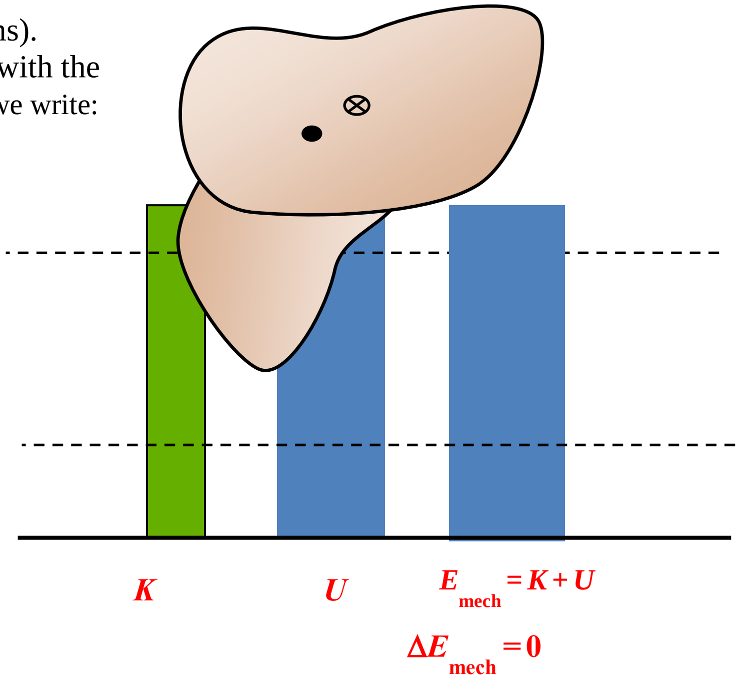
Moments of Inertia of Various Bodies



CONSERVATION OF ENERGY

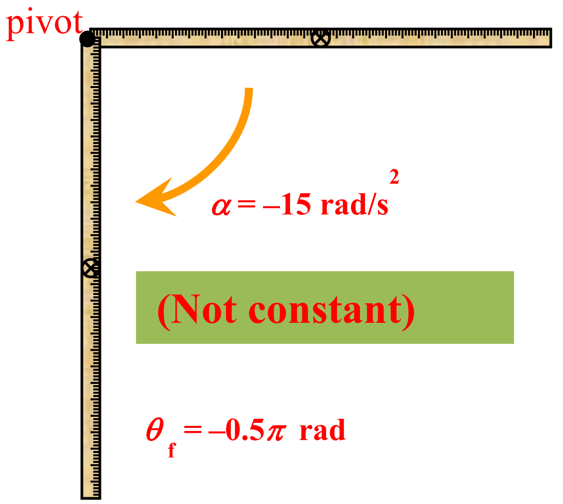
As usual, energy is conserved (in frictionless systems). If, however, a horizontal axis of rotation does not coincide with the centre of mass, the object's potential energy will vary. So we write:

$$E_{mech} = K_{rot} + U_g = \tfrac{1}{2} I\omega^2 + Mgy_{CM}$$



ROTATION ABOUT A FIXED AXIS

Example: A 70 g meter stick pivoted freely at one end is released from a horizontal position. At what speed does the far end swing through its lowest position?



$\theta_i = 0$ rad

$\omega_i = 0$ rad/s

$\theta_f = -0.5\pi$ rad

$\omega_f = ?$

$$\omega_f^2 = \omega_i^2 + 2\alpha\Delta\theta$$

$$\therefore\ \omega_f^2 = 0 + 2(-15)(-0.5\pi)$$

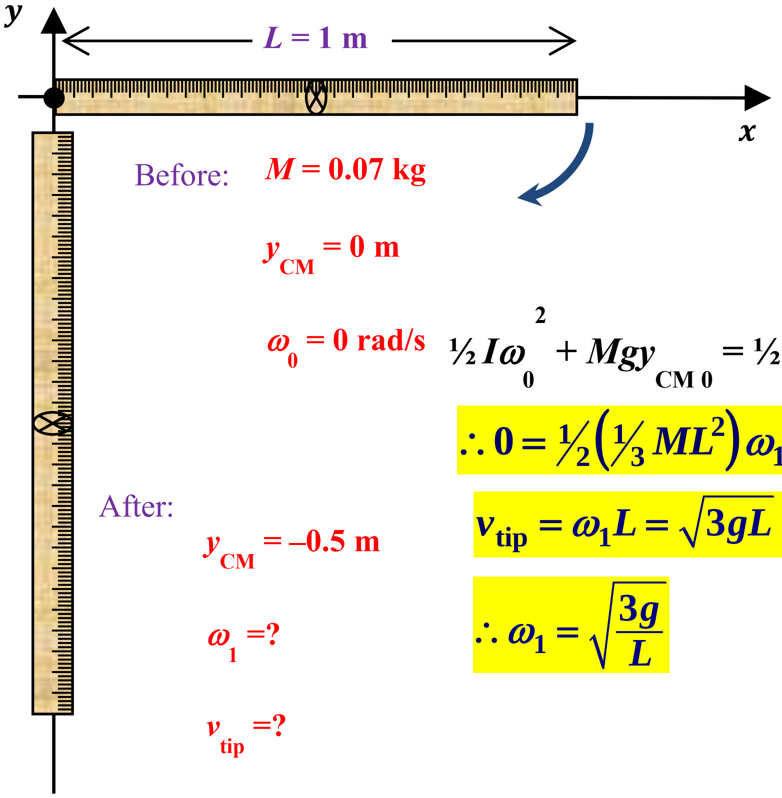
$$\therefore\ \omega_f = \sqrt{15\pi} = -6.8\ \text{rad/s}$$

$$v_t = \omega r$$

$$\therefore\ v_t = -6.8 \times 1 = \underline{6.8\ \text{m/s}}$$ ✗ !

You can NOT use rotational kinematics to solve this problem! Why not?

The same example: **The correct solution.**



Before: $M = 0.07$ kg

$y_{CM} = 0$ m

$\omega_0 = 0$ rad/s

After: $y_{CM} = -0.5$ m

$\omega_1 = ?$

$v_{tip} = ?$

$$\tfrac{1}{2} I\omega_0^2 + Mgy_{CM\,0} = \tfrac{1}{2} I\omega_1^2 + Mgy_{CM\,1}$$

$$\therefore\ 0 = \tfrac{1}{2}\left(\tfrac{1}{3}ML^2\right)\omega_1^2 + Mg\left(-\tfrac{1}{2}L\right)$$

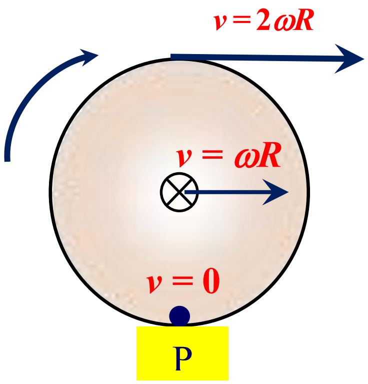
$$v_{tip} = \omega_1 L = \sqrt{3gL} = 5.42\ \text{m/s}$$

$$\therefore\ \omega_1 = \sqrt{\frac{3g}{L}}$$

❖ KINETIC ENERGY OF A ROLLING OBJECT

If we regard P as an instantaneous axis of rotation, the object's motion simplifies to one of pure rotation, and thus its kinetic energy is given by:

$$K = K_{\text{rot about P}} = \tfrac{1}{2} I_P \omega^2$$



Using the parallel axis theorem

$$I_P = (I_{CM} + MR^2)$$

$$K = \tfrac{1}{2} I_{CM}\omega^2 + \tfrac{1}{2} M(R\omega)^2$$

$$\therefore\ K = \tfrac{1}{2} I_{CM}\omega^2 + \tfrac{1}{2} M(v_{CM})^2$$

$$K = K_{\text{rot}} + K_{\text{CM}}$$

❖ THE GREAT DOWNHILL RACE

$$K_f = U_i$$

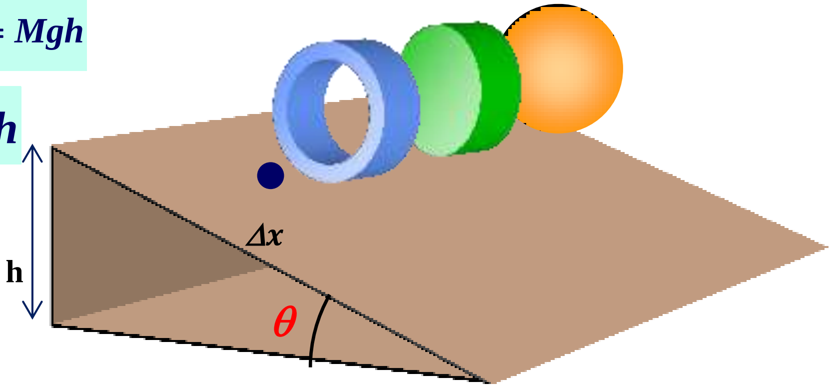
$$\therefore\ \tfrac{1}{2} I_{CM}\omega^2 + \tfrac{1}{2} M(v_{CM})^2 = Mgh$$

$$I_{CM} = cMR^2 \quad \text{and} \quad \omega = \frac{v_{CM}}{R}$$

$$\therefore\ \tfrac{1}{2} cMR^2\left(\frac{v_{CM}}{R}\right)^2 + \tfrac{1}{2} M\left(v_{CM}\right)^2 = Mgh$$

$$\therefore\ \tfrac{1}{2} M\left(1+c\right)\left(v_{CM}\right)^2 = Mgh$$

$$\therefore\ v_{CM} = \sqrt{\frac{2gh}{(1+c)}}$$



where c is a number $c \leq 1$.
that depends on the shape of the body.

I.e. The actual values of M and R do not feature, but *where the mass is situated* is of critical importance.

$$v_{CM} = \sqrt{\frac{2gh}{(1+c)}}$$

$$v_{CM}^2 = 0 + 2a\Delta x \quad \text{where } \Delta x = h/\sin\theta$$

$$\therefore\ a_{CM} = \frac{\left(v_{CM}\right)^2}{2\Delta x} = \frac{2gh/(1+c)}{2h/\sin\theta} = \frac{g\sin\theta}{1+c}$$

$$\therefore\ a_{CM} = \frac{a_{\text{particle}}}{1+c}$$

I.e. The acceleration of a rolling body is less than that of a particle by a factor which depends on the body's moment of inertia.

For a given value of c, the speed after descending a distance h is independent of the body's mass M and radius R. Hence all uniform solid cylinders have the same speed at the bottom, regardless of their mass and radii.

The values of c tell us that the order of finish for uniform bodies will be as follows: (1) any solid sphere $= \frac{2}{5}$, (2) any solid cylinder $c = \frac{1}{2}$, (3) any thin-walled, hollow sphere $c = \frac{2}{3}$, and (4) any thin-walled, hollow cylinder $c = 1$.

Small-c bodies always beat large-c bodies because less of their kinetic energy is tied up in rotation and so more is available for translation.

VECTOR DESCRIPTION OF ROTATIONAL MOTION

Using only "clockwise" and "counter clockwise" is the rotational analogue of using "backwards" and "forwards" in rectilinear kinematics. A more general handling of rotational motion requires vector quantities.

The vector associated with a rotational quantity…

- has magnitude equal to the magnitude of that quantity;
- has direction as given by the right-hand rule.

E.g. The angular velocity vector $\boldsymbol{\omega}$ of this anticlockwise-turning disc points I the positive z-direction.

THE CROSS PRODUCT

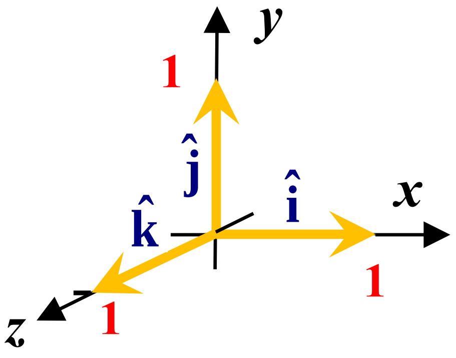
The magnitude of the torque exerted by force $\boldsymbol{F}$ applied at displacement $\boldsymbol{r}$ from the turning point is:

$$\tau \equiv rFsin\phi$$

Once again, the quantity $\boldsymbol{rF\ sin\phi}$ is the product of two vectors, $\boldsymbol{r}$ and $\boldsymbol{F}$, at an angle ϕ to each other. This time, however, we use the *orthogonal* components to determine the cross product of the vectors: $r \times F$.

$$\boldsymbol{i \times i = j \times j = k \times k = 0}$$

$$\begin{array}{ll} \boldsymbol{i \times j = k} & \boldsymbol{j \times i = -k} \\ \boldsymbol{j \times k = i} & \boldsymbol{k \times j = -i} \\ \boldsymbol{k \times i = j} & \boldsymbol{i \times k = -j} \end{array}$$



$$r \times F = rfsin\varphi$$

NOTE

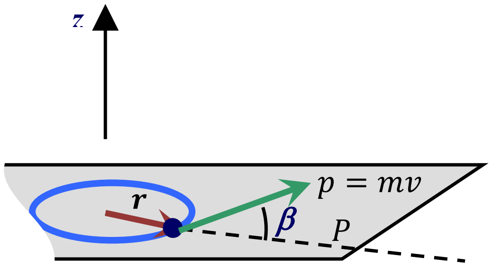
- The more orthogonal the vectors, the greater the cross product; the more parallel, the smaller…
  - Since it is a vector quantity, the cross product is also known as the vector product.
  - $\boldsymbol{A \times (B + C) = A \times B + A \times C}$.
  - Derivative of a cross product:

$$\frac{\boldsymbol{d}}{\boldsymbol{dt}}(\boldsymbol{r \times p}) = \frac{\boldsymbol{dr}}{\boldsymbol{dt}} \times \boldsymbol{p} + \boldsymbol{r} \times \frac{\boldsymbol{dp}}{\boldsymbol{dt}}$$

❖ ANGULAR MOMENTUM

We have shown that in circular motion (where $\mathbf{v}_t$ and $\boldsymbol{r}$ are perpendicular) a particle has angular momentum $\boldsymbol{L} = \boldsymbol{m r v_t}$.

$$mv_t = p \;\; and \;\; \therefore L = rp$$



More generally (allowing for r and p to be at *any* angle β)…

$\boldsymbol{L} = \boldsymbol{r} \times \boldsymbol{p} \quad = (mrv \sin\beta,$ direction from RH rule)

$$\frac{d\boldsymbol{L}}{dt} = \frac{d}{dt}(\boldsymbol{r} \times \boldsymbol{p}) = \frac{d\boldsymbol{r}}{dt} \times \boldsymbol{p} + \boldsymbol{r} \times \frac{d\boldsymbol{p}}{dt} = \boldsymbol{v} \times \boldsymbol{p} + \boldsymbol{r} \times \boldsymbol{F}_{net}$$

$\dfrac{d\boldsymbol{L}}{dt} = \boldsymbol{\tau}_{net}$ **(Cf. in linear motion:** $\boldsymbol{F}_{net} = \dfrac{d\boldsymbol{p}}{dt}$ **)**

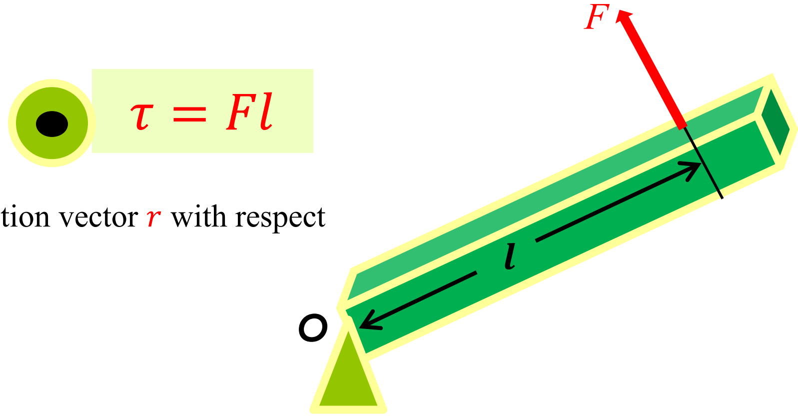
❖ ROTATIONAL MOMENTUM & ENERGY

Summary of corresponding quantities and relationships

Linear	**Rotational**
$K_{CM} = \frac{1}{2} M v_{CM}^2$	$K_{rot} = \frac{1}{2} I \omega^2$
$P = M v_{CM}$	$L = I\omega$ (around an axis of symmetry)
$F_{net} = \dfrac{dP}{dt}$	$\boldsymbol{\tau}_{net} = \dfrac{d\boldsymbol{L}}{dt}$
Linear momentum, *P*, is conserved if there is no net force	Angular momentum, *L* , is conserved if there is no net torque

Lecture 10: Dynamics of rotational motion

- **Torque: In general, for a force of magnitude F whose line of action is a perpendicular distance l from O. The torque is $\tau = Fl$**

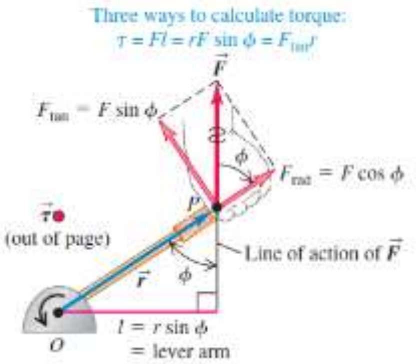


The force F applied at a point P described by a position vector r with respect to the chosen point O.

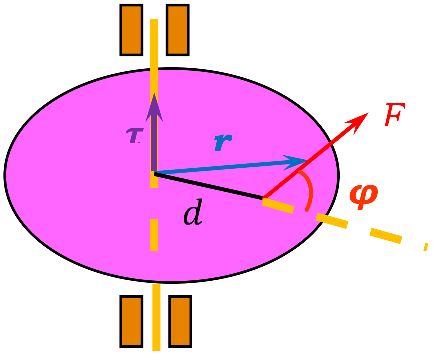
☐ There are three ways to calculate the torque of this force:

1. Find the lever arm l and use $\tau = Fl$.
2. Determine the angle φ between the vectors r and F; the lever arm is $r\sin\varphi$, so $\tau = rf\sin\varphi$.
3. Represent F in terms of a radial component F_{rad} along the direction of r and a tangential component F_{tan} at right angles, perpendicular to r.

Then $F_{tan} = F\sin\varphi$, and $\tau = r(F\sin\varphi) = F_{tan}r$. The torque caused by F_{rad} equal zero because its lever arm w.r.t O is zero.



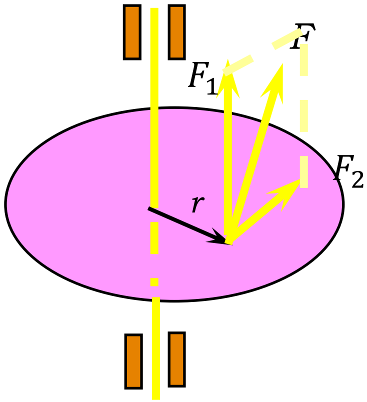
① F is located in the plane perpendicular to the axis



$$\boxed{\tau = r \times F}$$

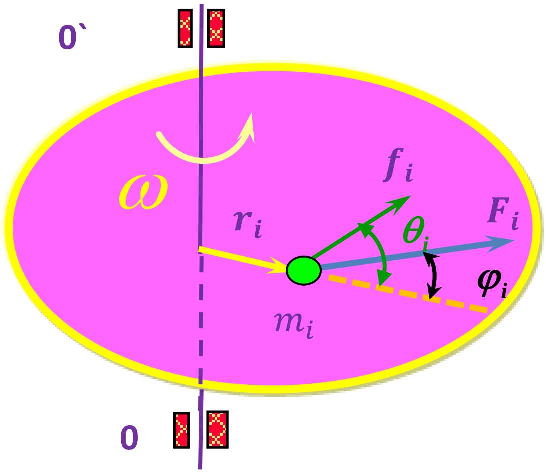
② F is not placed in the plane perpendicular to the axis

$$\tau = r \times F$$
$$= r \times (F_1 + F_2)$$



- $r \times F_1$ Has No effect at all to rotate the rigid body around the axis.
- Only $r \times F_2$ is useful for the rotation around the axis.

❖ The law of rotation



Using Newton's second law to m_i
The radial component of force passes through the axis and can't cause the body to rotate.

Tangential component

$$F_i sin\varphi_i + f_i sin\theta_i = m_i a_{it}$$
$$= (m_i r_i)\alpha$$

- Multiply both sides by r_i

$$\underbrace{F_i r_i \sin\varphi_i}_{\text{External Torque}} + \underbrace{f_i r_i \sin\theta_i}_{\text{Internal Torque}} = (m_i {r_i}^2)\alpha$$

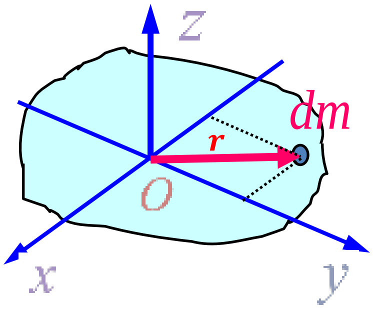
- **For entire rigid body**

$$\underbrace{\sum_i F_i r_i sin\varphi_i}_{\text{Resultant External Torque}} + \underbrace{\sum_i f_i r_i sin\theta_i}_{\text{Resultant internal Torque} = 0} = \sum_i \underbrace{(m_i {r_i}^2)\alpha}_{\text{Rotational inertia } I}$$

$$\boxed{\therefore \text{Law of Rotation is } \tau = I\alpha = I\frac{d\omega}{dt}}$$

③ Perpendicular-axis theorem

Suppose a thin board is located in x y- plane



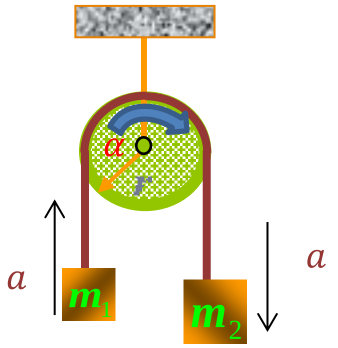
$$I_z = \int r^2 dm$$
$$= \int (x^2 + y^2)\, dm$$
$$= \int x^2 dm + \int y^2 dm$$

$$\boxed{\therefore\ I_z = I_x + I_y \quad \text{-- Perpendicular-axis theorem}}$$

❖ Application of the Law of Rotation

Example 1:

$\boldsymbol{m_1, m_2}$ are fastened together by a weightless rope across a fixed pulley $(\boldsymbol{m_1 < m_2})$
The pulley has mass m and radius r. There is a fractional torque τ_r exerting on the axis. The rope does not slip over the pulley.
Calculate the Acceleration of the blocks and the tension of the rope.



Solution

According to Newton's Second Law

$$m_1 : T_1 - m_1 g = m_1 a \qquad \text{(a)}$$

$$m_2 : m_2 g - T_2 = m_2 a \qquad \text{(b)}$$

According to Rotational Law

$$Pulley: T_2 r - T_1 r - \tau_r = I\alpha$$

$Where\ a = r\alpha\ and\ \ I = \frac{1}{2} m r^2$

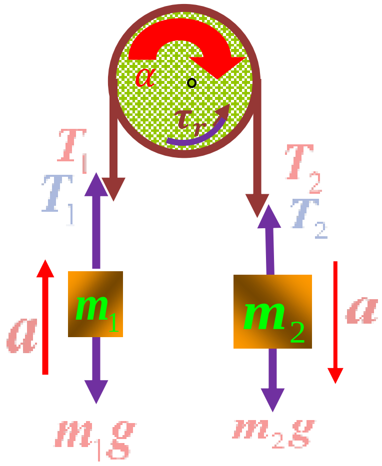
We get

$$a = \frac{(m_2 - m_1)g - \frac{\tau_r}{r}}{m_1 + m_2 + \frac{m}{2}}$$

From eqs. (a) and (b)

$$T_1 = m_1(g + a)$$

$$T_2 = m_2(g - a)$$



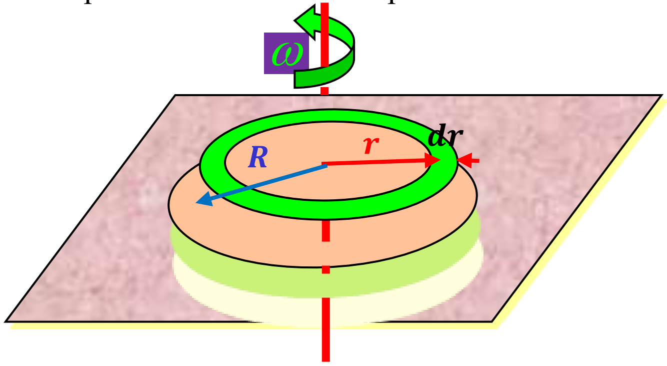
Example 2: A uniform circular plate with mass m and radius R is placed on a roughly horizontal plane. At the beginning, the plate rotates with angular speed ω_0 around the axis across the center of it. Suppose the fractional coefficient between them is μ. Calculate how many times does the plate rotate before it stops?

Solution:

- Select mass unit

$dm = \sigma ds$

$\sigma = \frac{m}{s} = \frac{m}{\pi R^2}$ and $ds = 2\pi r dr$



The friction exerts on dm

$df_r = \mu g dm = 2\pi\mu g\sigma r dr$

Frictional torque correspondingly

$$\boldsymbol{d\tau_r = r df_r = 2\pi\mu g\sigma r^2 dr}$$

$$\boldsymbol{\tau_r = \int d\tau_r}$$

$$= \int_0^R \boldsymbol{2\pi\mu g\sigma r^2 dr} = \frac{2}{3}\mu m g R$$

According to the law of rotation $\tau = I\frac{d\omega}{dt}$

We have

$$-\frac{2}{3}\mu mgR = \frac{1}{2}mR^2\frac{d\omega}{dt}$$

Simplifying and integrating using separation of variables

$$-\frac{2}{3}\mu g\int_0^t dt = \frac{1}{2}R\int_{\omega_0}^{0} d\omega$$

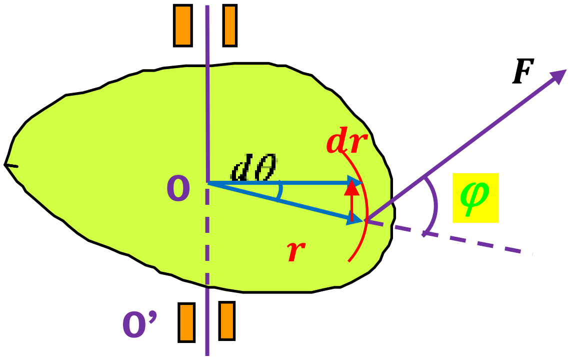
$$-\frac{2}{3}\mu gt = -\frac{1}{2}R\omega_0$$

$$\therefore t = \frac{3}{4}\frac{R}{\mu g}\omega_0$$

Q) How many revolutions does the plate rotate before it stops?

❖ Kinetic Energy and Work in Rotational Motion

1. The work done by torque



As $\boldsymbol{dW = F.dr}$

$= Frsin\varphi d\theta = \tau\, d\theta$

The rigid rotating from $\theta_1 \rightarrow \theta_2$

The work done by the torque $\boldsymbol{W = \int_{\theta_1}^{\theta_2} \tau d\theta}$

2. Kinetic Energy of rotation

The kinetic energy of rotating mass Δm_i is

$$\Delta KE_i = \frac{1}{2}\Delta m_i {v_i}^2$$

But $v = r\omega$

$$\therefore \Delta KE_i = \frac{1}{2}\Delta m_i {r_i}^2\omega^2$$

For the entire rigid body

$$KE = \sum_i \Delta KE_i = \sum_i \frac{1}{2}(\Delta m_i {r_i}^2)\omega^2$$

But $\boldsymbol{I = \sum_i (m_i {r_i}^2)}$

Therefore the kinetic energy of rotating rigid body is

$$KE = \frac{1}{2}I\,\omega^2$$

❖ The theorem of KE of a Rigid Body rotating about a fixed axis

Since $\boldsymbol{\tau = I\alpha = I\frac{d\omega}{dt}}$ **and the work**

$$\boldsymbol{dW = \tau d\theta = I\frac{d\omega}{dt}\, d\theta = I\frac{d\theta}{dt}\, d\omega = I\omega\, d\omega}$$

Integrating both sides

$$\int dW = \int_{\theta_1}^{\theta_2} \tau\, d\theta = \int_{\omega_1}^{\omega_2} I\omega\, d\omega$$

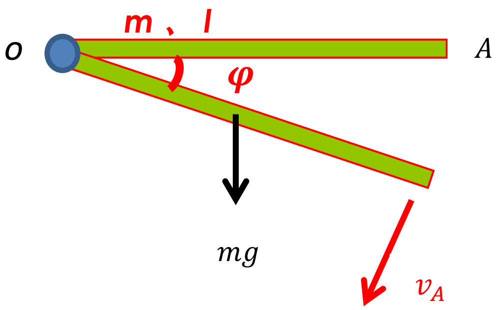
$$\boxed{\therefore W = \frac{1}{2} I {\omega_2}^2 - \frac{1}{2} I {\omega_1}^2}$$

❖ Gravitational Potential Energy of Rigid Body

Can be seen as the GPE of a particle located on the center of mass with the same mass of m

Example: A uniform thin rod of mass m and length l, one end is fixed.
Find its $v_A = ?$, $a_A = ?$ at point A as it rotate angular θ from horizontal line?

Solution:



As $\quad W = \frac{1}{2} I {\omega_2}^2 - \frac{1}{2} I {\omega_1}^2$

$\int_0^{\varphi} W d\varphi = \int_0^{\varphi} mg \frac{1}{2} \, l \cos\varphi d\varphi = \frac{1}{2} I \omega^2 - 0$

But $I = \frac{1}{3} m l^2$ and $v_A = l\,\omega$

$\frac{1}{2} mgl \sin\varphi = \frac{1}{2} I \omega^2 = \frac{1}{2}\frac{1}{3} m l^2 \omega^2$

$\omega^2 = \frac{3g}{l} \sin\varphi$

$\frac{{v_A}^2}{l^2} = \frac{3g}{l} \sin\varphi$

$\therefore v_A = \sqrt{3gl \sin\varphi}$

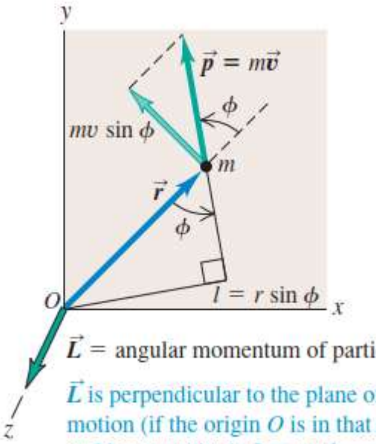
$a_n = \frac{{v_A}^2}{l} = 3g \sin\varphi \qquad a_t = \frac{dv_A}{dt} = \frac{3}{2} g \cos\varphi$

❖ Angular Momentum

A vector quantity denoted as $\boldsymbol{L}$ Its relationship to linear momentum $\boldsymbol{p}$ is exactly the same as the relationship of torque to force $\boldsymbol{\tau} = \boldsymbol{r} \times \boldsymbol{F}$, For a particle with constant mass m, velocity $\boldsymbol{v}$, momentum $\boldsymbol{p} = m\boldsymbol{v}$ and position vector $\boldsymbol{r}$ relative to the origin O of an inertial frame, we define angular momentum as

$$\boldsymbol{L} = \boldsymbol{r} \times \boldsymbol{p} = \boldsymbol{r} \times m\boldsymbol{v}$$

The value of $\boldsymbol{L}$ depends on the choice of origin O, since it involves the particle's position vector relative to O. The units of angular momentum are $kg.m^2/s$



$\vec{L}$ = angular momentum of particle

$\vec{L}$ is perpendicular to the plane of motion (if the origin O is in that plane) and has magnitude $L = mvl$.

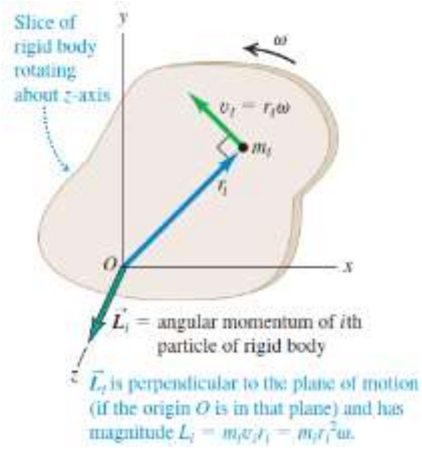
a particle moves in the xy-plane; its position vector $\boldsymbol{r}$ and momentum $\boldsymbol{p} = m\boldsymbol{v}$ are shown. The angular momentum vector $\boldsymbol{L}$ is perpendicular to the xy-plane. The right-hand rule for vector products shows that its direction is along the $+z-axis$ and its magnitude is

$$L = mvr \sin\phi = mvl$$

The rate of change of angular momentum of a particle equals the torque of the net force acting on it.

$$\frac{dL}{dt} = r \times F = \tau$$

❖ Angular Momentum of a rigid Body



The angular momentum of a particle of mass Δm_i in a rigid body rotating at angular speed ω.

$$\Delta L_i = \Delta m_i v_i r_i$$

$$= \Delta m_i (r_i\omega) r_i$$

$$\therefore \Delta L_i = \Delta m_i {r_i}^2 \omega$$

The total angular momentum of the rigid body is the sum of the angular momentum of all the particles

$$L = \sum_i L_i = \sum_i (\Delta m_i {r_i}^2)\omega = I\omega$$

❖ Angular momentum theorem

$$\tau = \frac{dL}{dt} \rightarrow \tau dt = dL = d(I\omega)$$

Integrating both sides

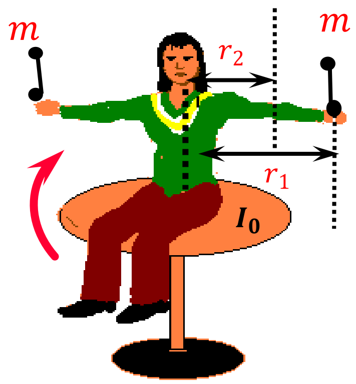
$$\int_{t_1}^{t_2} \tau dt = \int_{L_1}^{L_2} dL = \int_{I_1\omega_1}^{I_2\omega_2} d(I\omega)$$

$$\boxed{\therefore \int_{t_1}^{t_2} \tau\, dt = I_2\omega_2 - I_1\omega_1}$$

$\int_{t_1}^{t_2} \tau dt$ **Is the impulse momentum**

❖ Conservation Law of Angular Momentum

When the *net external torque* acting on a system is *zero*, the *total angular momentum* of the system is *constant* (*conserved*).



Since $\tau = \frac{dL}{dt}$

If $\tau = 0$ then $\frac{dL}{dt} = 0$ $\therefore L = I\omega = constant$, i.e. the Angular Momentum is conserved.

Example 1: The rotational inertia of a person and turntable is I_0. The mass of dumbbell is m. Their rotating angular speed is ω_0 and the rotating radius of m is r_1 at the beginning.

Calculate the angular speed and the increment of the mechanical energy when the arms of the person contracts from r_1 to r_2.

Solution: Person + turntable + dumbbells = system

Resultant external torque is zero. So its angular momentum is conserved. So

$$L_1 = L_2$$

$$[I_0 + (mr_1{}^2)_{left\ arm} + (mr_1{}^2)_{right\ arm}]\omega_0 = [I_0 + (mr_2{}^2)_{left\ arm} + (mr_2{}^2)_{right\ arm}]\omega$$

$$\therefore \omega = \left[\frac{I_0 + 2mr_1{}^2}{I_0 + mr_2{}^2}\right]\omega_0$$

The increment of the mechanical energy

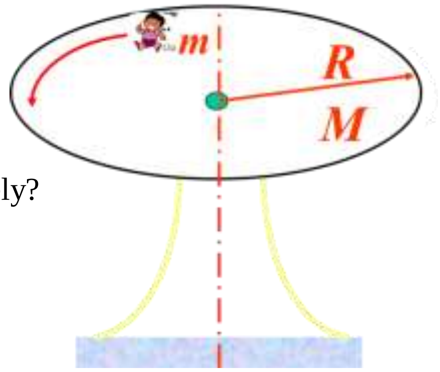
$$\Delta KE = KE - KE_0$$

$$= \frac{1}{2}(I_0 + mr_2{}^2)\omega^2 - \frac{1}{2}(I_0 + 2mr_1{}^2)\omega_0{}^2$$

- Solve for $m = 4\ kg,\ I_0 = 3\ kg.m^2,\ r_1 = 1\ m,\ r_2 = 0.5\ m,\ \omega_0 = 6\ rev/s$

Example 2: A round platform has mass of M and radius of R. It can rotates around a vertical axis through its center. Suppose all resistant force can be neglected.

A girl with mass of m stands on the edge of the platform. At the beginning, the platform and the girl are at rest. If the girl runs one revolution, how much degree does the girl and the platform rounds relative to the ground, respectively?



Solution: Resultant external torque of system = 0

Angular momentum is conserved : $L = L_0$

Let I, I_M be rotational inertia of the girl and the platform, respectively.

ω, Ω be angular speed of the girl and the platform relative to the ground, respectively

$$L = L_0$$

$$\therefore I\omega = I_M\Omega$$

$$I = mR^2, \qquad I_M = \frac{1}{2}MR^2$$

$$mR^2\omega = \frac{1}{2}MR^2\Omega$$

$$\therefore \Omega = \frac{2m}{M}\omega$$

The girl relative to the platform : $\omega' = \omega + \Omega = \frac{M+2m}{M}\omega$

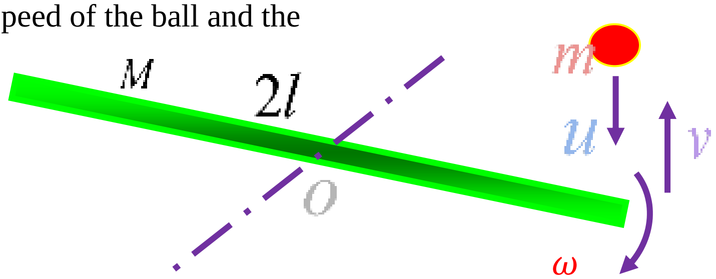
Let t refer to the time that the girl runs one revolution on the platform, then:

$$\int_0^t \omega' dt = \int_0^t \frac{M+2m}{M}\omega\, dt = 2\pi$$

The girl rounds relative to the ground : $\theta = \int_0^t \omega\, dt = \frac{2\pi M}{M+2m}$

The platform rounds relative to the ground : $\Theta = \int_0^t \Omega\, dt = \frac{2\pi m}{M+2m}$

Example 3: A uniformly thin rod has mass $M,$ and length $2l$. It can rotate in vertical plane around the horizontal axis through its mass center *O*. At the beginning, the rod is placed along the horizontal position. A small ball with mass *m* and speed *u* falls to one end of the rod. If the collision between the ball and the rod is elastic, find the speed of the ball and the angular speed of the rod after they collide each other.



Solution: As mg of the ball $<<$ the impulse force between the ball and the rod

So we can neglect mg during colliding.

Then the resultant external torque of the system with respect to *O* equals zero ($\tau = 0$.)

The angular momentum of the system is conserved.

$$mul = -mvl + I\omega$$

As the collision is elastic, then the mechanical energy of the system is conserved

$\frac{1}{2}mu^2 = \frac{1}{2}mv^2 + \frac{1}{2}I\omega^2$ (a)

And

$I = \frac{1}{12}m(2l)^2 = \frac{1}{3}ml^2$ (b)

Using equations. (a) and (b)

$$v = \frac{u(M-3m)}{M+3m}$$

$$\omega = \frac{6mu}{(M+3m)l}$$

Lecture 11: Equilibrium and Elasticity

❖ Conditions of equilibrium

For an object to be in translational equilibrium then

1. The vector sum of all the external forces acting on the body must be zero. $(\boldsymbol{F}_{net} = 0)$

   Translational equilibrium means that the linear momentum is constant,

$$\boldsymbol{p} = constant$$

$$\therefore \frac{d\boldsymbol{p}}{dt} = 0, \quad and \quad \therefore F_{net} = 0$$

For a body is in rotational equilibrium then

2. The vector sum of all the external torques the act on the body measured about any point must be zero. $(\boldsymbol{\tau}_{net} = 0)$

   Rotational equilibrium means that the angular momentum is constant,

$$\boldsymbol{L} = constant$$

$$\therefore \frac{d\boldsymbol{L}}{dt} = 0, \quad and \quad \therefore \boldsymbol{\tau}_{net} = 0$$

In component form the conditions of equilibrium are:

Balance of forces: $F_{net,x} = 0 \quad F_{net,y} = 0 \quad F_{net,z} = 0$

Balance of torques: $\tau_{net,x} = 0 \quad \tau_{net,y} = 0 \quad \tau_{net,z} = 0$

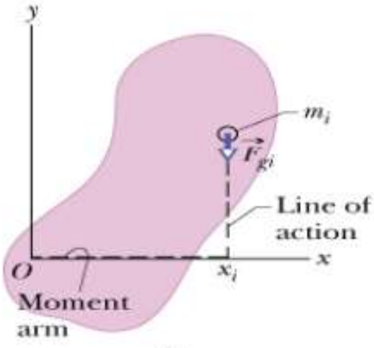
❖ The Center of Gravity

The gravitational force acting on an extended body is the vector sum of the gravitational forces acting on the individual elements of the body. The gravitational force F on a body effectively acts at a single point known as the center of gravity of the body.

We shall prove that if the acceleration of gravity $\boldsymbol{g}$ is the same for all the elements of the body then the center of gravity coincides with the center of mass.

- Consider the extended object of mass M shown in Fig. (a) Also shown the $i-th$ element of mass m_i. The gravitational force on m_i is equal to $m_i g_i$ where g_i is the acceleration of gravity in the vicinity of m_i. The net torque on m_i is

$$\tau_{net} = \textstyle\sum_i \tau_i = \sum_i F_{gi} x_i \qquad (1)$$



(a)

- If we replace the forces F_{gi} by the net gravitational force F_g acting at the center of gravity. The net torque Becomes

$$\tau_{net} = F_g x_{CG} = x_{CG} \textstyle\sum_i F_{gi} \qquad (2)$$

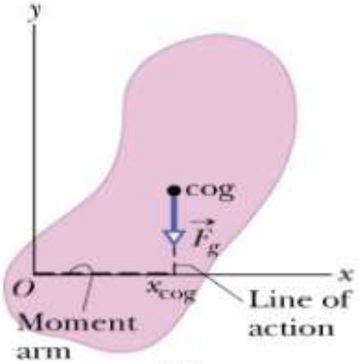
$$\tau_{net} = \textstyle\sum_i \tau_i = \sum_i F_{gi} x_i \qquad (1)$$

$$\tau_{net} = F_g x_{CG} = x_{CG} \textstyle\sum_i F_{gi} \qquad (2)$$

If we compare equation 1 with equation 2 we get:

$$x_{CG} \sum_i F_{gi} = \sum_i F_{gi} x_i$$

We substitute $m_i g_i$ for F_{gi} we get:



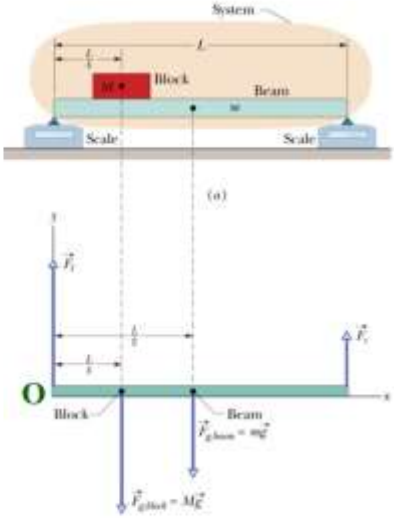
(b)

$$x_{CG}\sum_i m_i g_i = \sum_i m_i g_i x_i$$

If we set $g_i = g$ for all the elements we get:

$$x_{CG} = \frac{\sum_i m_i g x_i}{\sum_i m_i g} = \frac{\sum_i m_i x_i}{\sum_i m_i} = x_{CM}$$

Example1. A uniform beam of length L and mass $m = 1.8\ kg$ is at rest on two scales. A uniform block of mass $M = 2.7\ kg$ is at rest on the beam at a distance $L/4$ from its left end. Calculate the scales readings.



Solution:

$$\sum F_y = F_l + F_r - Mg - mg = 0 \qquad (1)$$

Choose to calculate the torque with respect to axis through the left end of the beam (point O)

$$\sum \tau_z = -\left(\frac{L}{4}\right)Mg - \left(\frac{1}{2}\right)mg + (L)F_r = 0 \qquad (2)$$

From equation (1)

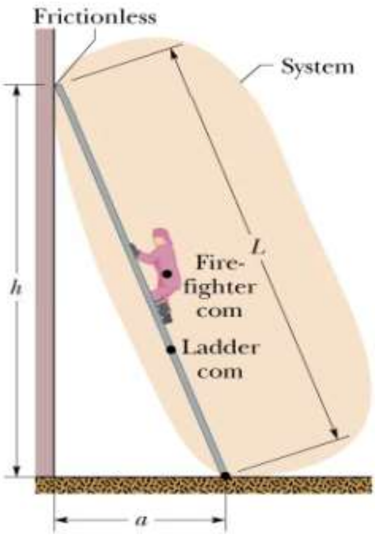
$$F_r = \frac{Mg}{4} + \frac{mg}{2} = \frac{2.7\times 9.8}{4} + \frac{1.8\times 9.8}{2} = 15.44 \approx 15\ \text{N}$$

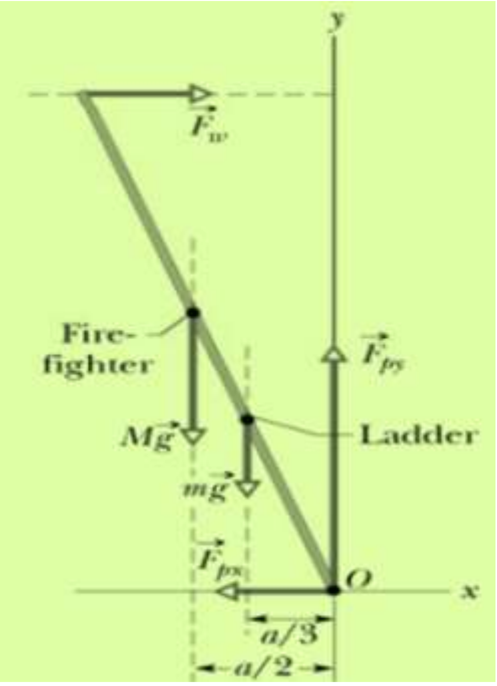
in equation (2)

$$F_l = Mg + mg - F_r = (2.7+1.8)\times 9.8 - 15.44 = 28.66\ \text{N}$$

Example 2. A ladder of length $L = 12\ m$ and mass $m = 45\ kg$ leans against a frictionless wall. The ladder's upper end is at a height $h = 9.3\ m$ above the pavement on which the lower end rests. The center of mass of the ladder is at $L/3$ from the lower end. A firefighter of mass $M = 72\ kg$ climbs half way up the ladder.

Find the forces exerted on the ladder by the wall and the pavement.





Solution:

Distance $a = \sqrt{L^2 - h^2} = 7.58\ m$

We take torques about an axis through point O.

$$\sum \tau_z = -hF_w + \frac{a}{3}mg + \frac{a}{2}Mg = 0$$

$$\therefore\ F_w = \frac{ga\left(\frac{M}{2} + \frac{m}{3}\right)}{h}$$

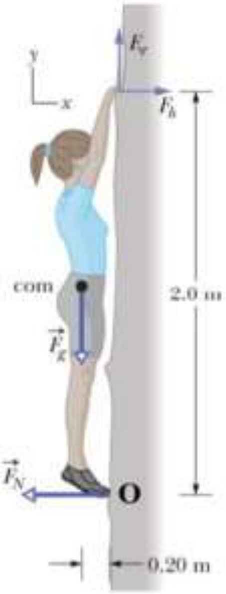
$$= \frac{9.8\times 7.58\times(72/2 + 45/3)}{9.3} = 407\ N$$

$\sum F_x = F_w - F_{px} = 0$, $\therefore F_{px}$ =407 N

$$\sum F_y = F_{py} - mg - Mg = 0$$

$$\therefore\ F_{py} = (72 + 45)\times 9.8 = 1146.6\ N$$

Example 3: Rock climber hangs by the crimp hold of one hand. Her feet touch the rock directly below her fingers. Assume that the force from the horizontal ledge supporting her fingers is equally shared by the four fingers. Calculate the horizontal and vertical components F_h and F_v of the force on each fingertip



Solution:

$$\sum F_x = -F_N + 4F_h = 0$$

$$\sum F_y = 4F_v - mg = 0$$

$$\therefore F_v = \frac{mg}{4} = \frac{70 \times 9.8}{4} = 171.5\ N$$

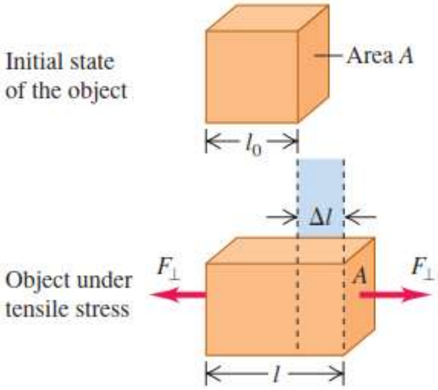
We calculate the net torque about an axis that is perpendicular to the page and passes through point O.

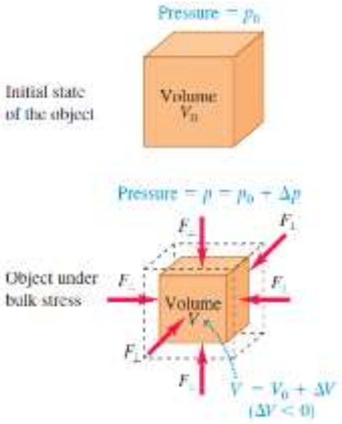
$$\sum \tau_z = (0)F_N + (0.2)mg - (2)(4F_h) + (0)(4F_v)$$

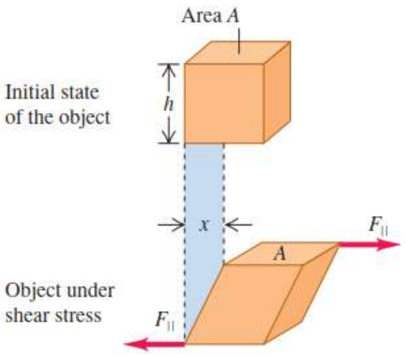
$$\therefore F_h = \frac{0.2 \times 70 \times 9.8}{4 \times 2} = 17.5\ N$$

❖ Elasticity

All "rigid" bodies are to some extend elastic, which means that we can change their dimensions slightly by pulling, pushing, twisting or compressing them.







In the three figures above we show the three ways in which a solid might change its dimensions under the action of external deforming forces.

❖ Stress, Strain, and Elastic Moduli

- **Tensile Stress.**
- **Bulk Stress.**
- **Shear Stress.**

For each type of deformation there are two relevant physical quantities:

- stress which characterizes the cause of the deformation
- strain which characterizes the effect of the deformation

Very often the two quantities are directly proportional to each other and the coefficient of proportionality is known as elastic modulus, defined as

$$\frac{\textbf{Stress}}{\textbf{Strain}} = \textbf{Elastic modulus}$$

which is also known as **Hooke's law**. In fact we have already seen an example of this law when elastic force of an ideal spring was considered, $\frac{F}{x} = k$

➢ Tensile and Compressive Stress and Strain

Tensile/compressive stress. For stretching/squeezing deformations the tensile/compressive stresses are defined as the ratio of the perpendicular force F to the cross-sectional area A.

$$\sigma = \frac{F}{A}$$

The SI unit of tensile stress is $Pascal(Pa) = N/m^2$.

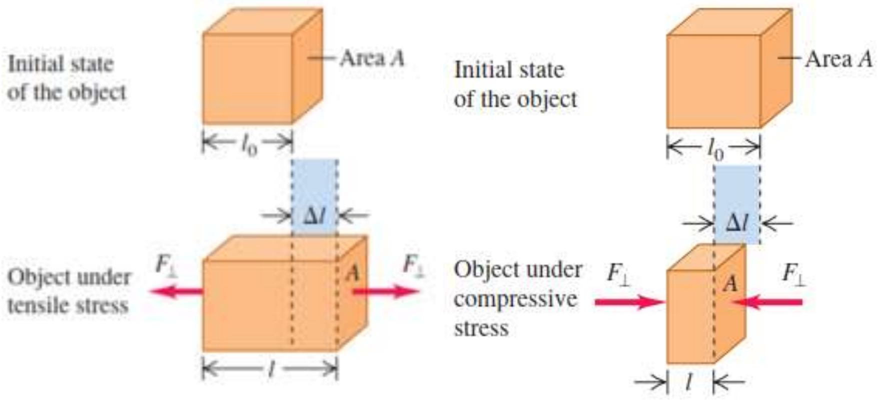
In the British system (psi) Pound per square inch is used

$$1 \text{ psi} = 6895 \text{ Pa} \quad \text{and} \quad 1 \text{ Pa} = 1.450 \times 10^{-4} \text{ psi}$$

The tensile strain of the object is equal to the fractional change in length

$$\text{Tensile strain} = \frac{l - l_0}{l_0} = \frac{\Delta l}{l_0}$$

➢ Stress, Strain, and Elastic Moduli



Experiment shows that for a sufficiently small tensile stress, stress and strain are proportional. The corresponding elastic modulus is called Young's modulus, denoted by ***Y***:

$$Y = \frac{\text{Tensile stress}}{\text{Tensile strain}} = \frac{F_\perp/A}{\Delta l/l_0} = \frac{F_\perp}{A}\frac{l_0}{\Delta l} \qquad \text{(Young's modulus)}$$

The units of Young's modulus are the same as those of stress: force per unit area.

Example: A steel 2.0 m long has a cross-sectional area of 0.30 cm^2. (Young's modulus Y for steel is $20 \times 10^{10} Pa$.) It is hung by one end from a support, and a 550 kg milling machine is hung from its other end. Determine the stress on the rod and the resulting strain and elongation.

Solution:

$$\text{Tensile stress} = \frac{F_\perp}{A} = \frac{(550\ \text{kg})(9.8\ \text{m/s}^2)}{3.0 \times 10^{-5}\ \text{m}^2} = 1.8 \times 10^8\ \text{Pa}$$

$$\text{Strain} = \frac{\Delta l}{l_0} = \frac{\text{Stress}}{Y} = \frac{1.8 \times 10^8\ \text{Pa}}{20 \times 10^{10}\ \text{Pa}} = 9.0 \times 10^{-4}$$

$$\text{Elongation} = \Delta l = (\text{Strain}) \times l_0$$
$$= (9.0 \times 10^{-4})(2.0\ \text{m}) = 0.0018\ \text{m} = 1.8\ \text{mm}$$

➢ Bulk Stress and Strain

Bulk stress is sometimes called Hydraulic stress

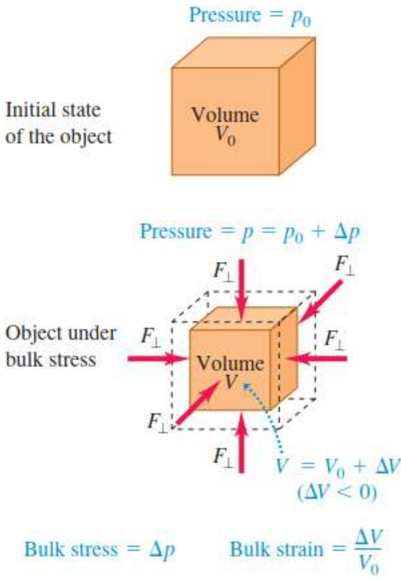
The stress is now a uniform pressure on all sides, and the resulting deformation is a volume change. We use the terms bulk stress (or volume stress) and bulk strain (or volume strain) to describe these quantities.

The force per unit area that the fluid exerts on the surface of an immersed object is called the pressure (p) in the fluid:

$$p = \frac{F}{A}$$

Pressure has the same units as stress. Also in common use is the atmosphere, abbreviated atm. One atmosphere is the average pressure of the earth’s atmosphere at sea level.

$$1\ \text{atmosphere} = 1\ \text{atm} = 1.013 \times 10^5\ \text{Pa} = 14.7\ \text{lb/in.}^2$$



The stress is fluid pressure $p = F/A$
Strain is a dimensionless ratio $\Delta V/V_0$
The modulus is called the bulk modulus B

- When the pressure on a body changes by a small amount Δp from p_0 to $p_0 + \Delta p$, and the resulting bulk strain is $\Delta V/V_0$, Hooke’s law takes the form

$$B = \frac{\text{Bulk stress}}{\text{Bulk strain}} = -\frac{\Delta p}{\Delta V/V_0} \qquad \text{(bulk modulus)}$$

We include a minus sign in this equation because an increase of pressure always causes a decrease in volume. In other words, if Δp is positive, ΔV is positive. The bulk modulus B itself is positive quantity.

- The reciprocal of the bulk modulus is called the compressibility and is denoted by k.

$$k = \frac{1}{B} = -\frac{\Delta V/V_0}{\Delta p} = -\frac{1}{V_0}\frac{\Delta V}{\Delta p}$$

Compressibility is the fractional decrease in volume $-\Delta V/V_0$, per unit increase Δp in pressure. The units of compressibility are those of *reciprocal pressure*, Pa^{-1}or atm^{-1}

Example: A hydraulic press contains $V_0 = 0.25\ m^3$ ($250\ L$) of oil. Find the decrease in the volume ΔV of the oil when it is subjected to a pressure increase $\Delta p = 1.6 \times 10^7$ Pa (about $160\ atm$). The bulk modulus of the oil is $B = 5.0 \times 10^9\ Pa$ (about $5.0 \times 10^4 atm$) and its compressibility is $k = 1/B = 20 \times 10^{-6} atm^{-1}$.

Solution:

From the definition of bulk modulus we have

$$\Delta V = -\frac{\Delta p V_0}{B} = -\frac{1.6\ \times 10^7 \times 0.25}{5.0\ \times 10^9} = -8 \times 10^{-4} m^3$$

or alternatively from the definition of bulk compressibility

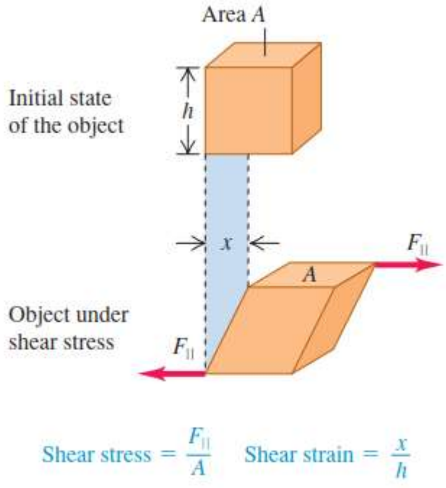
$$\Delta V = -\Delta p V_0 k = -(160\ atm)(0.25\ m^3)(20 \times 10^{-6} atm^{-1})$$
$$= -8 \times 10^{-4} m^3$$

❖ Shear Stress and Strain

In addition to tensile/compressive deformations due to perpendicular component of the applied force, a parallel component of the applied force can cause a deformation.

$$\text{Shear stress} = \frac{F_\parallel}{A}$$

$$\text{Shear strain} = \frac{x}{h}$$



- The corresponding elastic modulus is called the shear modulus, denoted by S:

$$S = \frac{\text{Shear stress}}{\text{Shear strain}} = \frac{F_\parallel/A}{x/h} = \frac{F_\parallel}{A}\frac{h}{x} \qquad \text{(shear modulus)}$$

Example: Suppose a brass base plate of an outdoor sculpture experiences shear force in an earthquake. The vertically oriented square plate is x = 0.5 cm thick. What is the force exerted on each of its edges is the resulting displacement is Δx = 0.16 mm?

Solution:

$$\text{Shear strain} = \frac{x}{h} = \frac{1.6 \times 10^{-4}\ \text{m}}{0.80\ \text{m}} = 2.0 \times 10^{-4}$$

$$\text{Shear stress} = (\text{Shear strain}) \times S$$
$$= (2.0 \times 10^{-4})(3.5 \times 10^{10}\ \text{Pa}) = 7.0 \times 10^6\ \text{Pa}$$

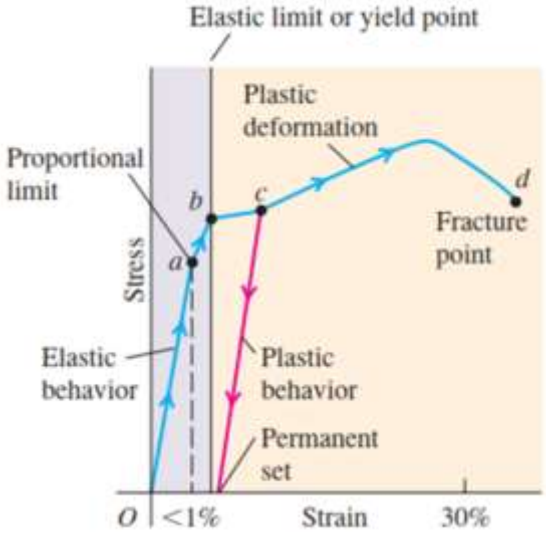
$$F_\parallel = (\text{Shear stress}) \times A$$
$$= (7.0 \times 10^6\ \text{Pa})(0.80\ \text{m})(0.0050\ \text{m}) = 2.8 \times 10^4\ \text{N}$$

❖ Elasticity and Plasticity

For larger deformations the linear dependence of stress as functions of strain (Hook's Law) does not hold.

The first portion (from O to a) the relation between stress and strain is linear (Hook's law). a is the proportional limit (From *a to b*), stress and strain are no longer proportional, and Hooke's law is not obeyed. In region Ob the material shows elastic behavior. Point b, the end of this region, is called the yield point; the stress at the yield point is called the elastic limit.

Further increase of load beyond *c* produces a large increase in strain for a relatively small increase in stress, until a point d is reached at which fracture takes place. The behavior of the material from b to d is called plastic deformation.



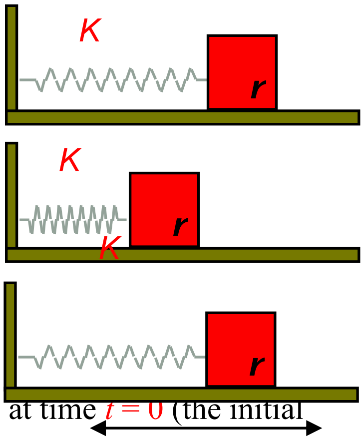
- **ductile materials**: are those which could show plastic deformation. Such materials can be actually drawn or bent or rolled before it reaches its fracture point. Examples are copper, aluminum etc.…
- **Brittle materials** are those which fracture before field point. Examples are glasses, Ceramics, wood etc.…

Ductile Material	Brittle Material
Solid materials that can undergo substantial plastic deformation prior to fracture are called ductile materials.	Solid materials that exhibit negligible plastic deformation are called brittle materials.
Percentage elongation of the ductile materials before fracture under tensile testing is higher.	Percentage elongation of the brittle materials before fracture under tensile testing is very less.
Ductile materials fail gradually by neck formation under the action of external tensile loading.	Brittle materials fail by sudden fracture (without any warning such as necking).
Energy absorbed by ductile materials before fracture under tensile testing is more.	Brittle materials absorb very small energy before fracture.
Various metal forming operations (such as rolling, forging, drawing, bending, etc.) can be performed on ductile materials.	Forming operations cannot be easily performed on brittle materials. For example, brittle material cannot be drawn into wire.
Ductile materials show longer life when subjected to fatigue loading.	Brittle materials fail faster when subjected to fatigue loading.

Examples of ductile material:	Examples of brittle material:
• Mild steel • Aluminum • Copper • Rubber • Most plastics	• Cast iron • Ceramics such as glass, cement, concrete, etc. • Stone • Ice

Lecture 12: Periodic Motion

- **Periodic Motion**: Any motion that repeats itself at regular intervals is called periodic motion.
  - Examples: circular motion, oscillatory motion
  - We know that if we stretch a spring with a mass on the end and let it go, the mass will oscillate back and forth (if there is no friction).



➢ This oscillation is called Simple Harmonic Motion(SHM)

The position of the object is

$$x(t) = A\cos(\omega t + \varphi)$$

- Angular frequency ω: determined by the inertia of the moving objects and the restoring force acting on it. SI: rad/s.
- Amplitude A: The maximum distance of displacement to the equilibrium point.
- phase $(\omega t + \varphi)$, phase angle (constant) (φ)

The value of A and (φ) depend on the displacement and velocity of the particle at time $t = 0$ (the initial conditions)

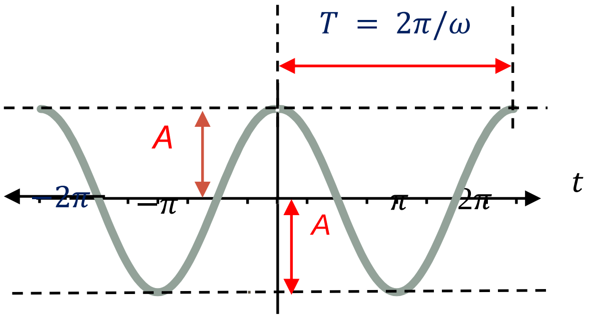
Period T: the time for one complete oscillation (or cycle);

$$A\cos(\omega t + \varphi) = A\cos[\omega(t + T) + \varphi]$$

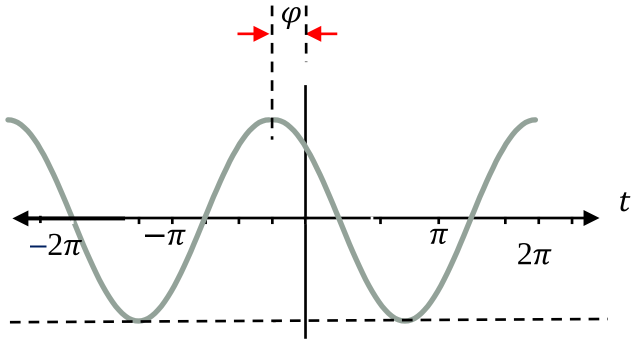
$$\Rightarrow T = \frac{2\pi}{\omega}$$

Frequency f number of oscillations that are completed each second.

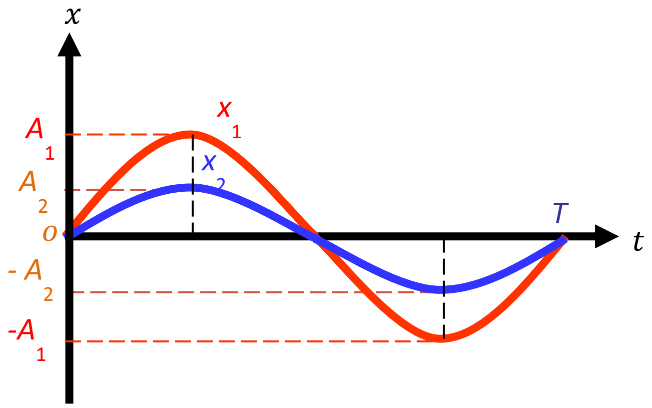
$$f = \frac{1}{T} \qquad \therefore \omega = \frac{2\pi}{T} = 2\pi f$$



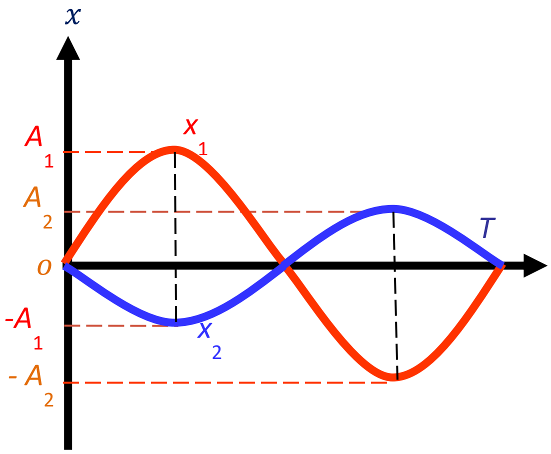
$$x = A\cos(\omega t)$$



$$x = A\cos(\omega t + \varphi)$$



They are in phase



They have a phase difference of π

❖ Relations Among Position, Velocity, and Acceleration in Simple Harmonic Motion

- Relation between position (displacement) and velocity:

Position:

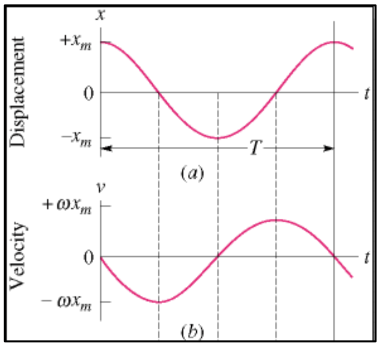
$$\boxed{x(t) = A\cos(\omega t + \varphi)}$$

Velocity is the derivative of the position:

$$\boxed{v(t) = \frac{dx}{dt} = -\omega A\sin(\omega t + \varphi)}$$

$$v(t) = \omega A\cos(\omega t + \varphi + \frac{\pi}{2})$$

- $v(t)$ leads $x(t)$ by $\pi/2$
- $v(t)$ is phase – shifted to the left from $x(t)$ by $\pi/2$.



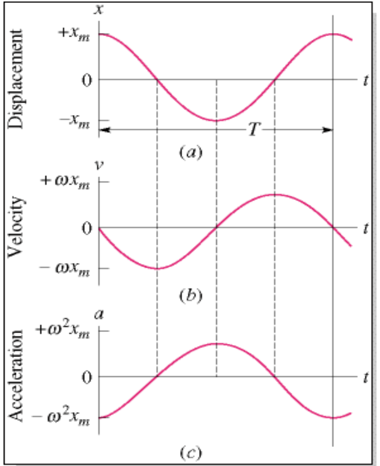
- $x(t)$ lags behind $v(t)$ by $\pi/2$
- $x(t)$ is phase – shifted to the right from $v(t)$ by $-\pi/2$

- Relation between acceleration and velocity and displacement

$$x(t) = A\cos(\omega t + \varphi) \qquad (1)$$

$$v(t) = \frac{dx}{dt} = -\omega A\sin(\omega t + \varphi) \qquad (2)$$

$$\boxed{a(t) = \frac{dv}{dt} = -\omega^2 A\cos(\omega t + \varphi) = -\omega^2 x(t) \qquad (3)}$$



$$= \omega^2 A\cos(\omega t + \varphi + \tfrac{\pi}{2})$$

- $a(t)$ *is phase – shifted to the left from* $x(t)$ *by* π

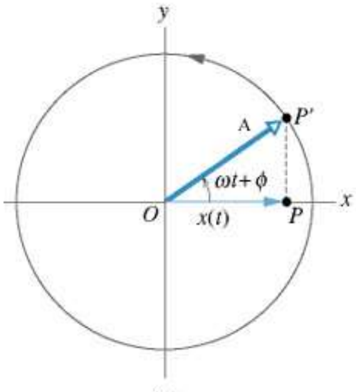
In SHM, the acceleration is proportional to the displacement but opposite in sign, and the two quantities are related by the square of the angular frequency.

❖ A Connection to Circular Motion

A reference particle P' moving in a reference circle of radius A with steady angular velocity ω. Its projection P on the x axis executes simple harmonic motion

$$x(t) = A\cos(\omega t + \varphi)$$

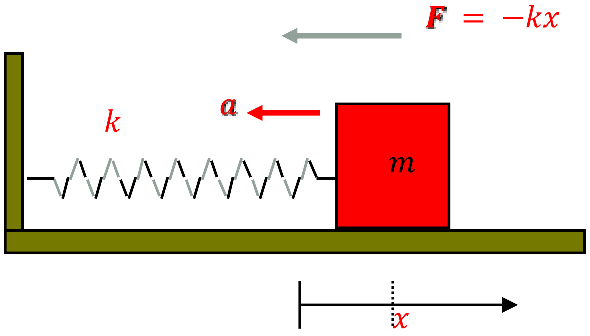
Simple harmonic motion is the projection of uniform circular motion on a diameter of the circle in which the latter motion occurs.



(a)

❖ Springs and Simple Harmonic Motion

The block–spring system, shown in the figure, forms a linear simple harmonic oscillator



Hooke's law $F = -kx$

Combining with Newton 2nd. Law $F = m\,a$

$$\therefore m\,a = -k\,x$$

$$a = \frac{d^2x}{dt^2} = -\frac{k}{m}\,x$$

Comparing with equation (3)

$$\omega^2 = \frac{k}{m} \qquad (4)$$

$$\frac{d^2x}{dt^2} + \omega^2 x = 0 \qquad (5)$$

The solution for differential equation (5) is

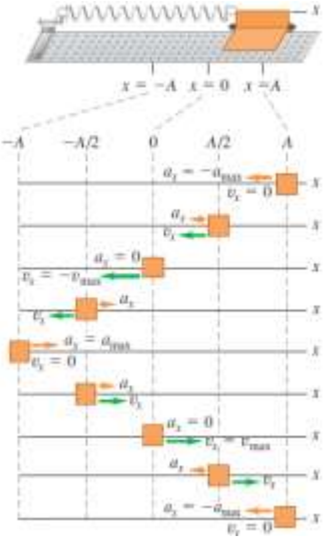
$$x(t) = A\cos(\omega t + \varphi)$$

With $\omega = \sqrt{\frac{k}{m}}$ and the period is $T = 2\pi\sqrt{\frac{m}{k}}$

The period of the motion is independent of the amplitude (no A in the equation)

For initial contentions: $t = 0;\ x = x_0,\ \ v = v_0$

$$A = \sqrt{x_0{}^2 + \frac{v_0{}^2}{\omega^2}}, \qquad \tan\varphi = -\frac{v_0}{\omega x_0}$$



- Another solution is

$$x = B\sin(\omega t) + C\cos(\omega t)$$

$$\frac{dx}{dt} = \omega B cos(\omega t) - \omega C\ sin(\omega t)$$

✓ $\frac{d^2x}{dt^2} = -\omega^2 B \sin(\omega t) - \omega^2\ C \cos(\omega t) = -\omega^2 x$

$x(t) = A\cos(\omega t + \varphi)$, is equivalent to $x(t) = B\sin(\omega t) + C\cos(\omega t)$

> Proof: $x(t) = A\cos(\omega t + \varphi) = A\cos(\omega t)\,cos\varphi - A sin(\omega t) sin\varphi$
> $= B\sin(\omega t) + C\cos(\omega t)$
> Where $B = -A\ sin\varphi$ *and* $C = A\ cos\varphi$

Example 1: A block whose mass m is 680 g is fastened to a spring whose spring constant k is 65 N/m. The block is pulled a distance x = 11 cm from its equilibrium position at $x = 0$ on a frictionless surface and released from rest at $t = 0$.

(a) What are the angular frequency, the frequency, and the period of the resulting motion?
(b) What is the amplitude of the oscillation?
(c) What is the maximum speed v_m of the oscillating block, and where is the block when it occurs?
(d) What is the magnitude a_m of the maximum acceleration of the block?
(e) What is the phase constant φ for the motion?
(f) What is the displacement function $x(t)$ for the spring–block system?

Solution:

(a) $\omega = \sqrt{\frac{k}{m}} = \sqrt{\frac{65}{0.680}} \approx 9.8\frac{rad}{s},\quad \omega = 2\pi f\ \rightarrow f = \frac{\omega}{2\pi} = \frac{9.8}{2\pi} = 1.56\ Hz,\quad T = \frac{1}{f} = \frac{1}{1.56} = 0.64\ s$

(b) $A = \sqrt{{x_0}^2 + \frac{{v_0}^2}{\omega^2}} = A = \sqrt{{x_0}^2 + \frac{0}{\omega^2}} = x_0 = 11\ cm$

(c) $v_{max} = \omega A = 9.8 \times 0.11 = 1.1\ m/s,$ at equilibrium point

(d) $a_{max} = \omega^2 A = (9.8)^2 \times 0.11 = 10.56\frac{m}{s^2}$

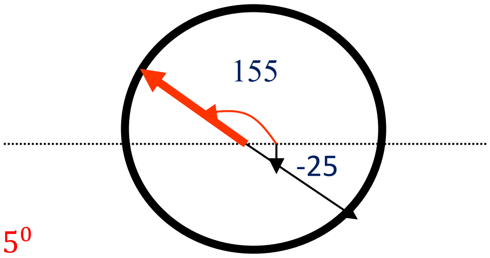
(e) $\varphi = tan^{-1}\left(\frac{v_0}{\omega x_0}\right) = tan^{-1}(0) = 0$

(f) $x(t) = A\cos(\omega t + \varphi) = 0.11\ cos(9.8\ t + 0) = 0.11\cos(9.8t)$

Example 2: At $t = 0$, the displacement $x(0)$ of the block in a linear oscillator is $-8.50\ cm$. The block's velocity $v(0)$ then is $-0.920\ m/s$, and its acceleration $a(0)\ is\ + 47.0\ m/s2$.

(a) What is the angular frequency w of this system?
(b) What are the phase constant *f* and amplitude *A*?

Solution:



(a) $a(t) = -\omega^2 x(t) \rightarrow \omega = \sqrt{-\frac{a}{x}} = \sqrt{\frac{47}{0.085}} = 23.5\ rad/s$

(b) $tan\varphi = -\frac{v_0}{\omega x_0} = -\frac{(-0.92)}{23.5\times(-0.085)} = -0.461\ \rightarrow \varphi = -25^0\ or\ 155^0$

The correct phase constant is 155^0

The amplitude is $A = \sqrt{{x_0}^2 + \frac{{v_0}^2}{\omega^2}} = 9.4\ cm$

❖ Vertical Springs

Choose the origin at equilibrium position

The net force acting on the body is

$$F_{net} = F_{spring} - mg$$
$$= kx - mg$$

$x = d - y$

$kd = mg$

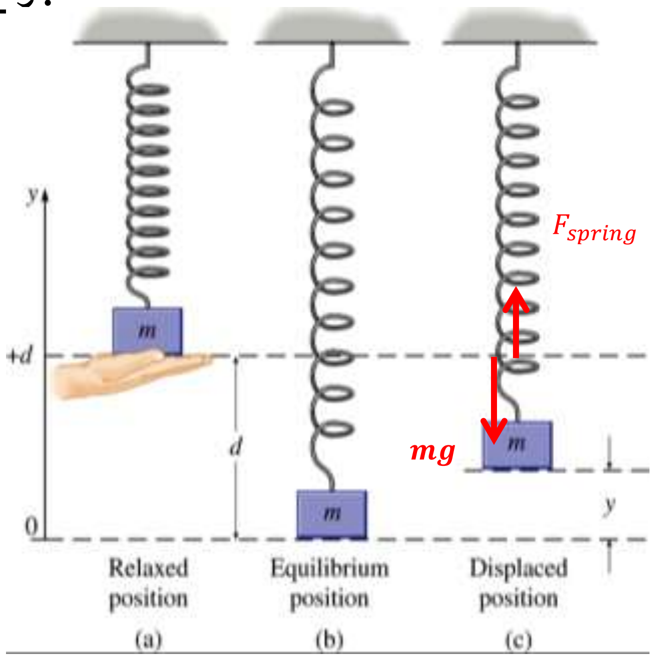
$$\therefore F_{net} = -ky$$

Therefore, the solution for this differential equation is

$y(t) = A\cos(\omega t + \varphi)$

With $\omega = \sqrt{\frac{k}{m}}$

Simple harmonic motion with equilibrium point at $y = 0$



❖ Energy in Simple Harmonic Motion

The kinetic energy of the body is $KE = \frac{1}{2}mv^2$ and the potential energy of the spring is$U = \frac{1}{2}kx^2$, There are no non-conservative forces that do work, so the total mechanical energy $E = KE + U$ is conserved:

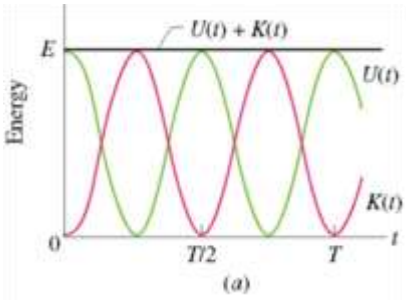
$$E = \frac{1}{2}mv^2 + \frac{1}{2}kx^2 = constant$$

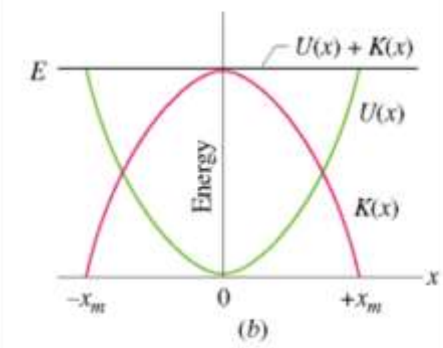
The total mechanical energy E is also directly related to the amplitude A of the motion. When the body reaches the point $x = A$ its maximum displacement from equilibrium, it momentarily stops as it turns back toward the equilibrium position. That is, when $x = A\ (or\ -A), v = 0$. At this point the energy is entirely potential, $E = \frac{1}{2}kA^2$ because E is constant; it is equal $\frac{1}{2}kA^2$ at any other point. Hence, the total mechanical energy is

$$\boxed{E = \frac{1}{2}mv^2 + \frac{1}{2}kx^2 = \frac{1}{2}kA^2 = constant}$$

Note

- ➢ The potential energy and the kinetic energy peak twice during every period

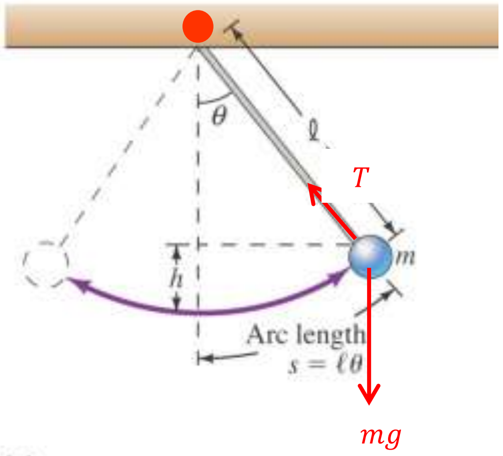




- ➢ The mechanical energy is conserved for a linear harmonic oscillator
- ➢ The dependence of energy on the square of the amplitude is typical of Simple Harmonic Motion

❖ The Simple Pendulum

A simple pendulum consists of a point-like mass m (called the *bob* of the pendulum) suspended from one end of an unstretchable, massless string of length l that is fixed at the other end



(a)

$$\tau = I\frac{d^2\theta}{dt^2}$$

$$-lmg\sin\theta = ml^2\frac{d^2\theta}{dt^2}$$

$$\frac{d^2\theta}{dt^2} + \frac{g}{l}\sin\theta = 0$$

For small θ, $\sin\theta \approx \theta$

$$\therefore \frac{d^2\theta}{dt^2} + \frac{g}{l}\theta = 0$$

Therefore, the solution for this differential equation is

$$\theta = \theta_0\cos(\omega t + \varphi)$$

With $\omega = \sqrt{\frac{g}{l}}$

The motion of a simple pendulum swinging through only small angles is approximately SHM.
The period of small-amplitude pendulum is independent of the amplitude --- the pendulum clock

$$T = 2\pi\sqrt{\frac{l}{g}}$$

The horizontal displacement $x = l\theta = A\cos(\omega t + \varphi)$, $\quad A = l\theta_0$

Example: Find the period and frequency of a simple pendulum $1\ m$ long at a location where $g = 9.8\ m/s^2$

$$T = 2\pi\sqrt{\frac{l}{g}} = T = 2\pi\sqrt{\frac{1}{9.8}} = 2.007\ s$$

$$f = \frac{1}{T} = \frac{1}{2.007} = 0.498\ Hz$$

❖ The energy of a simple pendulum:

$$\boxed{KE(\theta) = \frac{1}{2}I\left(\frac{d\theta}{dt}\right)^2 = \frac{1}{2}ml^2\left(\frac{d\theta}{dt}\right)^2}$$

$$U(\theta) = mgh = mgl(1 - \cos\theta)$$

For small θ, $\cos\theta = 1 - \frac{\theta^2}{2}$, so the potential energy becomes $U(\theta) = mgl\left(1 - 1 + \frac{\theta^2}{2}\right)$

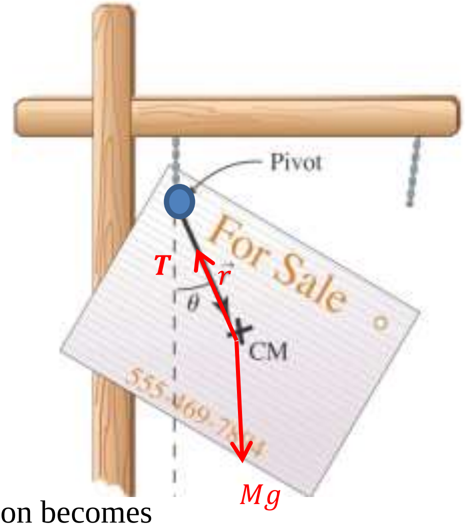
$$\boxed{U(\theta) = \frac{1}{2}mgl\theta^2}$$

The total energy is conserved

$$\boxed{E = KE + U = mgl(1 - \cos\theta_0) = mgh_0}$$

❖ The Physical Pendulum

Any object suspended and then displaced so the gravitational force does not run through the center of mass, can oscillate due to the torque. The motion is not simple harmonic because the torque is proportional to $\sin\theta$ rather than to θ itself. However, we can use the approximation for small angle $\sin\theta \approx \theta$. Then the motion is approximately simple harmonic.



For the example in the figure, the forces acting on the (For Sale) sign are its weight (mg) and the tension (T).

The torque

$$\tau = I\alpha = I\frac{d^2\theta}{dt^2}$$
$$= -rMg\sin\theta$$

Using the approximation $\sin\theta \approx \theta$ for small angle, the differential equation becomes

$$\frac{d^2\theta}{dt^2} + \frac{rMg}{I}\theta = 0$$

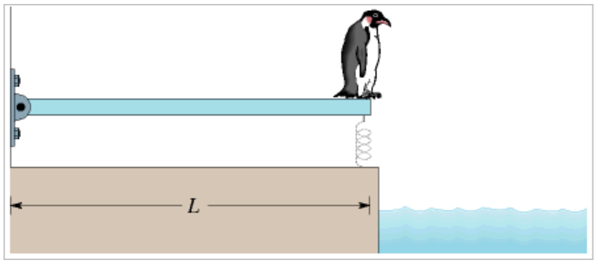
Then, its solution is

$$\theta = \theta_0 \cos(\omega t + \varphi)$$

With $\omega = \sqrt{\frac{Mgr}{I}}$, $\quad I = ml^2 = Mr^2 \quad \rightarrow \omega = \sqrt{\frac{g}{r}}$

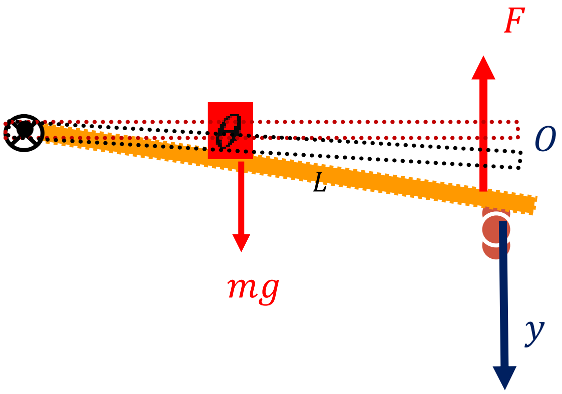
The period of oscillation is $T = 2\pi\sqrt{\frac{I}{Mgr}} = 2\pi\sqrt{\frac{r}{g}}$

The period of a physical pendulum is independent of its total mass—only how the mass is distributed matters.



Example: In Figure below, a penguin dives from a uniform board that is hinged at the left and attached to a spring at the right. The board has length $L = 2.0\ m$ and mass $m = 12\ kg$; the spring constant $k = 1300\ N/m$. When the penguin dives, it leaves the board and spring oscillating with small amplitude. Assume that the board is stiff enough not to bend, and find the period T of the oscillations.

Solution:



- Choose the equilibrium position as the origin

$$mg\frac{L}{2} = ky_0L$$

The torque is $\tau = I\frac{d^2\theta}{dt^2}$

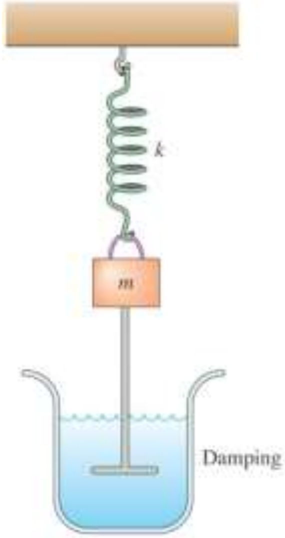
$$mg\frac{L}{2} - k(y_0 + y)L = I\frac{d^2\theta}{dt^2}, \qquad I = \frac{1}{3}ml^2 \ , \sin\theta \approx \theta$$

$$\Rightarrow \quad -kL^2\theta = \frac{1}{3}mL^2\frac{d^2\theta}{dt^2} \quad \Rightarrow \quad \frac{d^2\theta}{dt^2} + \frac{3k}{m}\theta = 0$$

$$\omega = \sqrt{\frac{3k}{m}}, \qquad T = 2\pi\sqrt{\frac{m}{3k}} = 2\pi\sqrt{\frac{12}{(3\times1300)}} = 0.35\ s$$

T is independent of the board's length.

❖ Damped Harmonic Motion



A pendulum does not go on swinging forever. Energy is gradually lost (because of air resistance) and the oscillations die away. This effect is called damping.

Look at drag force that is proportional to velocity; b is the damping coefficient:

$$\boldsymbol{F}_d = -b\boldsymbol{v} = -\boldsymbol{b}\frac{d\boldsymbol{x}}{dt}$$

Then the equation of motion is:

$$\sum F = -kx - bv = ma$$

$$-kx - b\frac{dx}{dt} = m\frac{d^2x}{dt^2}$$

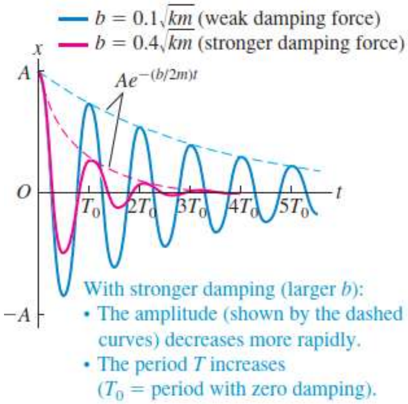
$$\frac{d^2x}{dt^2} + \frac{b}{m}\frac{dx}{dt} + \frac{k}{m}x = 0$$

$$\frac{d^2x}{dt^2} + 2\alpha\frac{dx}{dt} + {\omega_0}^2 x = 0$$

Where the damping factor $\alpha = \frac{b}{2m}$

If the damping force is relatively small, the motion is described by (solution for the 2nd order differential) equation is

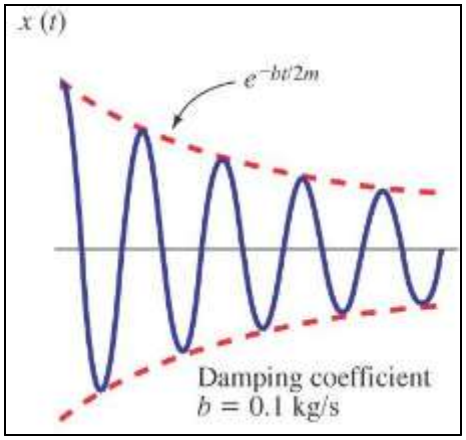
$$x(t) = Ae^{-\alpha t}\cos(w't + \varphi)$$



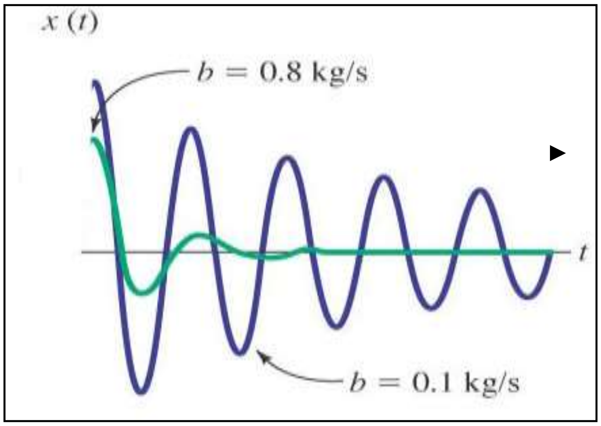
With stronger damping (larger b):
- The amplitude (shown by the dashed curves) decreases more rapidly.
- The period T increases (T_0 = period with zero damping).

The angular frequency of oscillation ω' is given by

$$\omega' = \sqrt{{\omega_0}^2 - \alpha^2} = \sqrt{\frac{k}{m} - \frac{b^2}{4m^2}}$$



$$x(t) = Ae^{-(b/2m)t}cos(\omega't + \varphi)$$



$$x(t) = Ae^{-t/2T}cos(\omega't + \varphi)$$

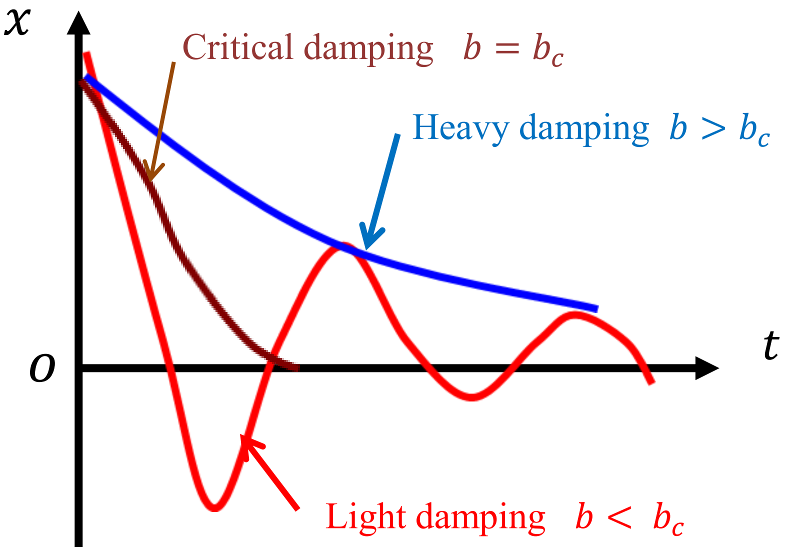
- The larger the value of the life time of the oscillation $T = m/b$, the slower the exponential.

The motion of damped oscillation differs from the undamped case in two ways.

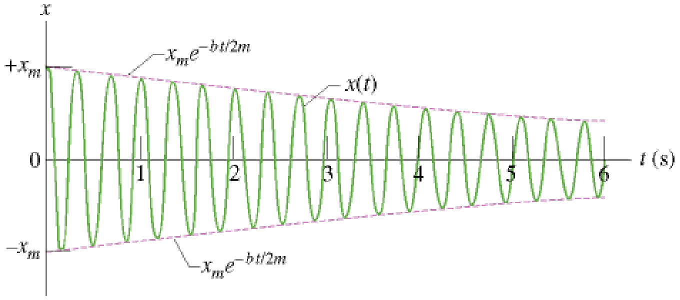
1. the amplitude $Ae^{-(b/2m)t}$ is not constant but decreases with time because of the decreasing exponential factor $e^{-(b/2m)t}$.
2. The angular frequency $\omega'$is no longer equal to $\omega = \sqrt{k/m}$ but is somewhat smaller. It becomes zero $(\omega' = 0)$ when b becomes so large that

$$\frac{k}{m} - \frac{b^2}{4m^2} = 0 \quad or \quad b = 2\sqrt{km} = b_c$$

Such condition is called critical damping



Example: For the damped oscillator:$m = 250\ g$, $k = 85\ N/m$, and $b = 70\ g/s$. (a) What is the period of the motion? (b) How long does it take for the amplitude of the damped oscillations to drop to half its initial value? (c) How long does it take for the mechanical energy to drop to one-half its initial value?



Solution:

a) $T = 2\pi\sqrt{\frac{m}{k}} = 2\pi\sqrt{\frac{0.25}{85}} = 0.34\ s$

a) $x(t) = A\ e^{-(b/2m)t} cos(\omega' t + \varphi)$

$$e^{-(b/2m)t} = \frac{1}{2} \longrightarrow \quad t = \frac{2m}{b} ln2 = 5\ s$$

b) $E \propto x^2 \propto e^{-bt/m}$

$$e^{-bt/m} = \frac{1}{2} \longrightarrow \quad t = \frac{m}{b} \ln 2 = 2.5\ s$$

❖ Driven Harmonic Motion

In damped harmonic motion, a mechanism such as friction dissipates or reduces the energy of an oscillating system, with the result that the amplitude of the motion decreases in time.

- We have considered the presence of a 'damping' force acting on an oscillator:

$$F_{\,damping} = -\,bv = -b\frac{dx}{dt}$$

- Now consider applying an external force:

$$F_{driving} = F_{max}\cos(\omega_d\,t)$$

- Every simple harmonic oscillator has a natural oscillation frequency (ω if undamped, ω' if underdamped)
- By applying $F_{driving} = F_{max}\cos(\omega_d\,t)$ we force the oscillator to oscillate at the frequency ω_d (can be anything, not necessarily ω or ω')
- We get a new equation of motion for $x(t)$:

$$ma = F_{\text{restoring}} + F_{\text{damping}} + F_{\text{driving}}$$

$$m\frac{d^2x}{dt^2} = -kx - b\frac{dx}{dt} + F_{\max}\cos\omega_{\text{d}}t$$

This has a general solution:

$$x = A_0e^{-\frac{bt}{2m}}cos\left(\sqrt{{\omega_0}^2 - \frac{b^2}{4m^2}}\,t + \varphi_0\right) + A\,cos(\omega_d t + \varphi)$$

We could work out an expression that shows how the amplitude A of the forced oscillation depends on the frequency of a sinusoidal driving force, with maximum value F_{max}

$$A = \frac{F_{max}}{\sqrt{(k - m{\omega_d}^2)^2 + (b\omega_d)^2}}$$

When $k - m{\omega_d}^2 = 0,$ so A has a maximum near $\omega_d = \sqrt{k/m}.$ The height of the curve at this point is proportional to $1/b$; the less damping, the higher the peak. At the low frequency extreme, when $\omega_d = 0,$ we get $A = F_{max}/k.$ This corresponds to constant force F_{max} and a constant displacement $A = F_{max}/k$ from equilibrium.